T1A

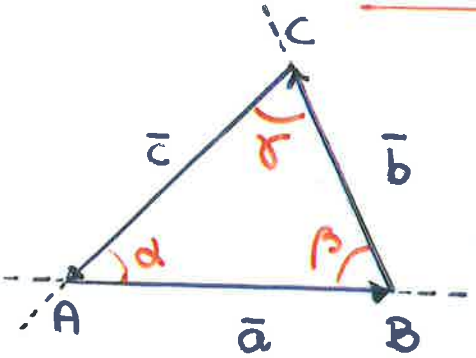
Esercizi sul calcolo vettoriale

1) Dim. che se $\bar a, \bar b, \bar c \in \mathbb{V}$ soddisfano alle condizioni

$$\vec a + \vec b + \vec c = \vec 0 \quad (\text{poligono chiuso})$$

allora:

$$\underline{\bar a \times \bar b = \bar b \times \bar c = \bar c \times \bar a} \qquad (1)$$



$$\bar a = B - A$$
$$\bar b = C - B$$
$$\bar c = A - C$$

$$\bar a + \bar b + \bar c = (B-A) + (C-B) + (A-C) \equiv \bar 0$$

molt. a sx vett. per $\bar b$: $\bar b \times \bar a + \underset{\bar 0}{\underset{\parallel}{\bar b \times \bar b}} + \bar b \times \bar c = \bar 0$

$$\Rightarrow \underline{\bar b \times \bar c} = -\bar b \times \bar a = \underline{\bar a \times \bar b}$$

molt. a sx vett. per $\bar c$: $\bar c \times \bar a + \bar c \times \bar b + \underset{\bar 0}{\underset{\parallel}{\bar c \times \bar c}} = \bar 0$

$$\Rightarrow \underline{\bar c \times \bar a} = -\bar c \times \bar b = \underline{\bar b \times \bar c}$$

2) Dim. il teorema dei seni:

Poiché vale (1) $\Rightarrow |\bar a \times \bar b| = |\bar b \times \bar c| = |\bar c \times \bar a|$

$$ab \underset{\text{sen}\,\beta}{\underset{\parallel}{\text{sen}(\pi-\beta)}} = bc \underset{\text{sen}\,\gamma}{\underset{\parallel}{\text{sen}(\pi-\gamma)}} = ac \underset{\text{sen}\,\alpha}{\underset{\parallel}{\text{sen}(\pi-\alpha)}}$$

$$\begin{aligned} a \,\text{sen}\,\beta &= c\,\text{sen}\,\gamma \\ b\,\text{sen}\,\gamma &= a\,\text{sen}\,\alpha \\ b\,\text{sen}\,\beta &= c\,\text{sen}\,\alpha \end{aligned} \quad \Rightarrow \quad \underline{\frac{a}{\text{sen}\,\gamma} = \frac{b}{\text{sen}\,\alpha} = \frac{c}{\text{sen}\,\beta}}$$

3) Dim. il th. di Carnot:

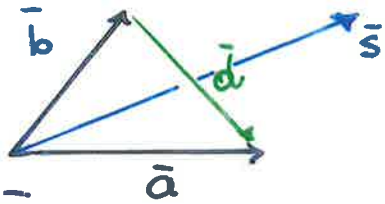
Da $\bar a + \bar b + \bar c = \bar 0 \quad \Rightarrow \quad \bar c = -(\bar a + \bar b)$

Allora $\bar{c}\cdot\bar{c} = -(\bar{a}+\bar{b})\cdot[-(\bar{a}+\bar{b})] = a^2+b^2+2\bar{a}\cdot b$

$$\Rightarrow \quad c^2 = a^2+b^2+2ab\cos(\pi-\beta) = a^2+b^2-2ab\cos\beta$$

$$c = \sqrt{a^2+b^2-2ab\cos\beta}$$

4) Det. C.N.S. affinché la somma e la differenza di 2 vettori $\bar{a}, \bar{b} \in \mathbb{V}$ siano vettori paralleli o ortogonali fra loro.



$\bar{a}, \bar{b} \neq \bar{0} \qquad \bar{s} = \bar{a}+\bar{b}\ , \ \bar{d} = \bar{a}-\bar{b}$

- $(\bar{a}+\bar{b}) \parallel (\bar{a}-\bar{b}) \iff (\bar{a}+\bar{b})\times(\bar{a}-\bar{b}) = \bar{0}$

$$\iff -\bar{a}\times\bar{b}+\bar{b}\times\bar{a} = \bar{0}$$

$$\iff \bar{a}\parallel\bar{b}$$

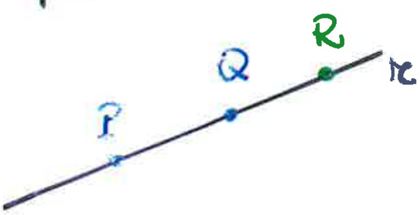
- $(\bar{a}+\bar{b}) \perp (\bar{a}-\bar{b}) \iff (\bar{a}+\bar{b})\cdot(\bar{a}-\bar{b}) = 0$

$$\iff a^2 - b^2 = 0 \quad \Rightarrow \quad a = \pm b.$$

se $a=b$ il parallelogramma è un rombo.

5) Scrivere l'eq. della retta in forma vettoriale.

In $Ox_1x_2x_3$ siano dati P, Q e la retta r passante per essi



$P = (x_1, x_2, x_3)$; $Q = (y_1, y_2, y_3)$

Dato un generico pto $R = (z_1, z_2, z_3)$

$R \in r$ se $(R-P)\parallel(R-Q)$ cioè:

$$\boxed{(R-P)\times(R-Q) = \bar{0}}$$

In coord. cart.

$$0 = \begin{vmatrix} \bar{e}_1 & \bar{e}_2 & \bar{e}_3 \\ z_1-x_1 & z_2-x_2 & z_3-x_3 \\ z_1-y_1 & z_2-y_2 & z_3-y_3 \end{vmatrix} \Rightarrow \frac{z_1-x_1}{z_1-y_1} = \frac{z_2-x_2}{z_2-y_2} = \frac{z_3-x_3}{z_3-y_3}$$

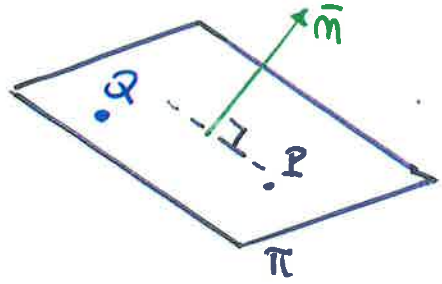
6) Scrivere l'eq. del piano passante per un punto (in forma vettoriale).

π : piano e $P \in \pi$ $P = (x_1, x_2, x_3)$ nel rif. $Ox_1x_2x_3$

$\bar{n}$: normale al piano π $\bar{n} = (\alpha_1, \alpha_2, \alpha_3)$ coseni direttori ($|\bar{n}| = 1$)

Dato $Q = (y_1, y_2, y_3)$, $Q \in \pi$ se $(Q-P) \perp \bar{n}$ cioè:

$$\boxed{(P-Q)\cdot\bar{n} = 0}$$



In coord. cartesiane:

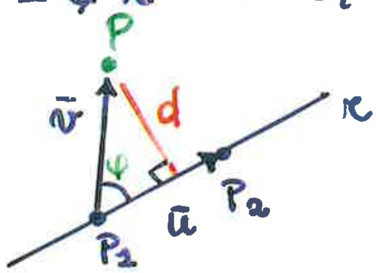
$$\sum_{i=1}^{3}(x_i - y_i)\vec{e}_i \cdot \sum_{J=1}^{3}\alpha_J\vec{e}_J = 0$$

$$\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 + \delta = 0$$

7) Scrivere in forma vettoriale la distanza di un punto da una retta.

$P \notin r$ e $P_1, P_2 \in r$ sia $\bar{u} = P_2 - P_1$

$\bar{v} = P - P_1$



$$d = v\,\mathrm{sen}\varphi = \frac{v\,u\,\mathrm{sen}\varphi}{u} = \boxed{\frac{|\bar{u}\times\bar{v}|}{|\bar{u}|}}$$

8) Risolvere l'eq. vettoriale:

$$\vec{x}\times\vec{a} = \bar{b} - \vec{x}$$

- se $\bar{a} = \bar{0}$ $\Rightarrow$ $\vec{x} = \vec{b}$
- se $\bar{a} \neq \bar{0}$

molt. vett. a dx per $\bar{a}$: $(\bar{x}\times\bar{a})\times\bar{a} = (\bar{b}-\bar{x})\times\bar{a}$

$\Rightarrow (\bar{x}\cdot\bar{a})\bar{a} - a^2\bar{x} = \bar{b}\times\bar{a} - \bar{b} + \bar{x}$

molt. scal. per $\bar{a}$: $\underbrace{(\bar{x}\times\bar{a})\cdot\bar{a}}_{= 0} = \bar{b}\cdot\bar{a} - \bar{x}\cdot\bar{a}$

$\Rightarrow \bar{x}\cdot\bar{a} = \bar{b}\cdot\bar{a}$

$\Rightarrow (1+a^2)\bar{x} = \bar{a}\times\bar{b} + \bar{b} + (\bar{b}\cdot\bar{a})\bar{a}$ da cui ricavo $\vec{x}$.