BARICENTRI

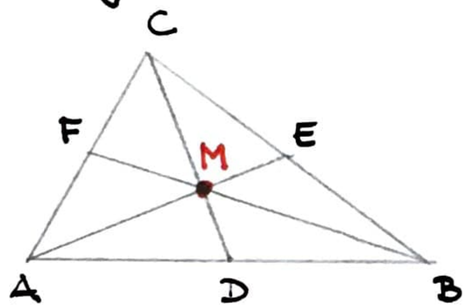
FIGURE OMOGENEE

1) asta AB : m, L $\Rightarrow$ G è il punto medio

2) quadrato ABCD : m, L $\Rightarrow$ G è l'incrocio delle diagonali

3) rettangolo ABCD : m, lati a, b $\Rightarrow$ G è " " "

4) triangolo ABC : m, lati a, b, c $\Rightarrow$ G è l'incontro delle **MEDIANE**

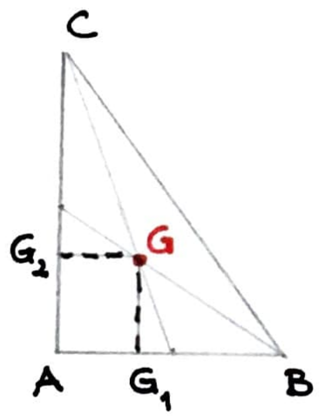


$\overline{MB} = 2\,\overline{MF}$

$\overline{CM} = 2\,\overline{MD}$

$\overline{AM} = 2\,\overline{ME}$

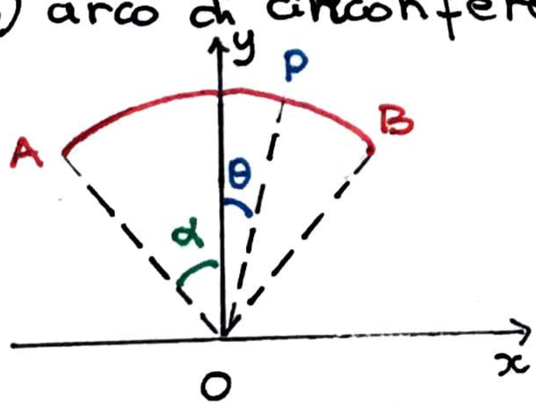
$M \equiv G$



$\overline{AG_1} = \frac{1}{3}\overline{AB}$

$\overline{AG_2} = \frac{1}{3}\overline{AC}$

5) arco di circonferenza (m, R) apertura 2α.



$m = \rho\,\widehat{AB} = \rho\, 2\alpha R$

Per simmetria $G \in Oy$.

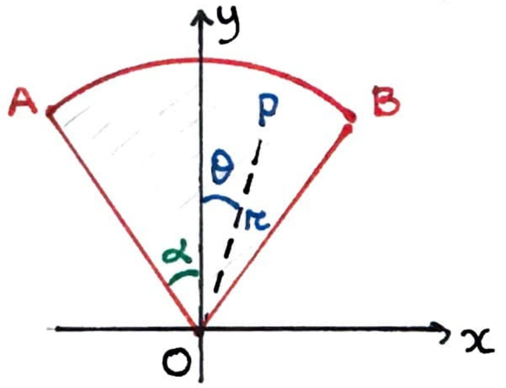
$y_G = \frac{1}{m}\int_B \rho\, y_P\, dB \qquad dB = R\,d\theta$

$$\boxed{y_G} = \frac{1}{\cancel{m}}\cdot\frac{\cancel{m}}{2\alpha R}\cdot\int_{-\alpha}^{\alpha} R^2\cos\theta\, d\theta = \frac{R}{2\alpha}\cdot 2\sin\alpha = \boxed{R\,\frac{\sin\alpha}{\alpha}}$$

In particolare se $\alpha = \frac{\pi}{2}$ $\quad y_G = \frac{2R}{\pi}$

$$\Rightarrow \begin{cases} G\left(0, R\frac{\sin\alpha}{\alpha}\right) & \text{per l'arco } AB \\ G\left(0, \frac{2R}{\pi}\right) & \text{per la semicirconferenza.} \end{cases}$$

6) settore circolare (m, R) apertura 2α.



$$m = \rho A = \rho \frac{2\alpha R . R}{2} = \rho \alpha R^2$$

Per simmetria $G \in Oy$.

$$y_G = \frac{1}{m}\int_{B} \rho\, y \, dB \qquad dB = r\, dr\, d\theta$$

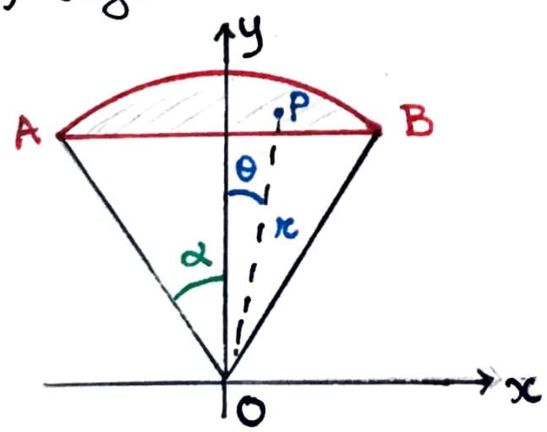
$$\boxed{y_G} = \frac{1}{\rho\alpha R^2}\cdot \rho \int_{-\alpha}^{\alpha}\int_0^R r\cos\theta \, r\, dr\, d\theta = \frac{1}{\alpha R^2}\int_{-\alpha}^{\alpha}\cos\theta\, d\theta \int_0^R r^2\, dr$$

$$= \frac{1}{\alpha R^2}\cdot 2\sin\alpha \cdot \frac{1}{3}R^3 = \boxed{\frac{2}{3}\,\frac{R\sin\alpha}{\alpha}}$$

In particolare se $\alpha = \frac{\pi}{2}$ $\quad y_G = \frac{4R}{3\pi}$

$$\Rightarrow \begin{cases} G\left(0, \frac{2}{3}R\frac{\sin\alpha}{\alpha}\right) & \text{per settore circolare } AB \\ G\left(0, \frac{4R}{3\pi}\right) & \text{per il semidisco} \end{cases}$$

7) segmento circolare ad una base: (m, R) apertura 2α.



Per simmetria $G \in Oy$.

Per determinare y_G posso applicare la proprietà distributiva

$m = m_{settore} - m_{triangolo}$

poiché ρ è costante $\Rightarrow$ si riduce al calcolo delle aree.

$A_1 =$ Area settore $= \frac{1}{2} \cdot 2\alpha R \cdot R = \alpha R^2$

$A_2 =$ Area triangolo $= \frac{b \cdot h}{2} = \frac{1}{2} \cdot 2R\sin\alpha \cdot R\cos\alpha = R^2 \sin\alpha\cos\alpha$

$A_{segmento} = R^2(\alpha - \sin\alpha\cos\alpha)$

Baricentro del settore $y_{G_1} = \frac{2}{3} R \frac{\sin\alpha}{\alpha}$

Baricentro del triangolo $y_{G_2} = \frac{2}{3} h = \frac{2}{3} \cdot R\cos\alpha$

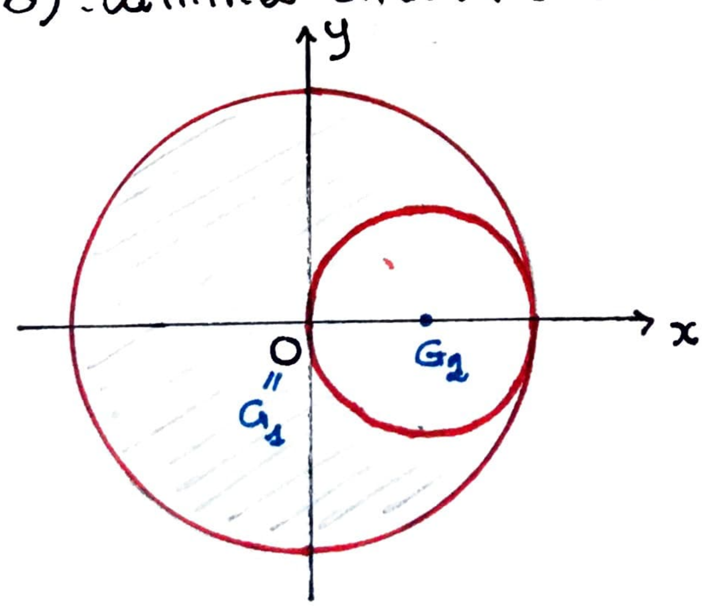
Per la proprietà distributiva:

$$\boxed{y_G} = \frac{m_1 y_{G_1} - m_2 y_{G_2}}{m_1 - m_2} = \frac{A_1 y_{G_1} - A_2 y_{G_2}}{A}$$

$$= \frac{\alpha \cancel{R^2} \cdot \frac{2}{3} R \frac{\sin\alpha}{\alpha} - \cancel{R^2}\sin\alpha\cos\alpha \cdot \frac{2}{3} R\cos\alpha}{\cancel{R^2}(\alpha - \sin\alpha\cos\alpha)}$$

$$= \frac{2}{3} \frac{\sin\alpha R(1 - \cos^2\alpha)}{\alpha - \sin\alpha\cos\alpha} = \boxed{\frac{2}{3} \frac{R\sin^3\alpha}{\alpha - \sin\alpha\cos\alpha}}$$

8) Lamina circolare con foro circolare : massa m



Disco : massa m_1, R

foro : massa m_2, $\frac{R}{2}$

Per simmetria $G \in Ox$.

Per calcolare x_G applico la proprietà distributiva

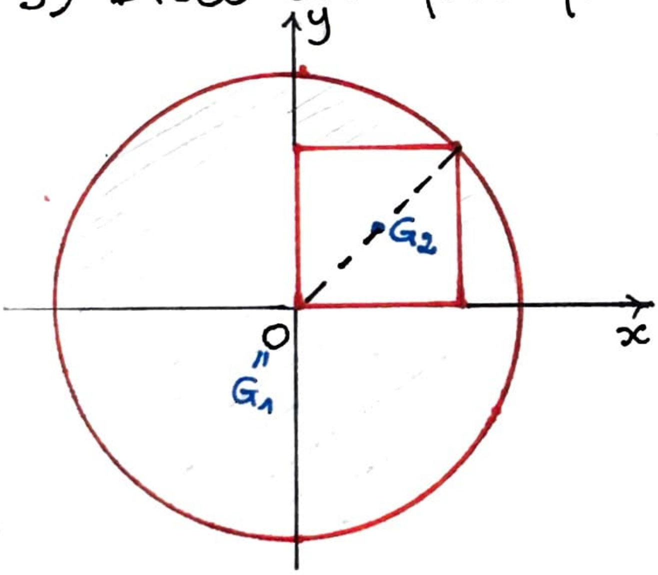
$$x_G = \frac{m_1 x_{G_1} - m_2 x_{G_2}}{m_1 - m_2} = \frac{A_1 x_{G_1} - A_2 x_{G_2}}{A}$$

$x_{G_1} \equiv 0$; $x_{G_2} = \frac{R}{2}$

$$A = \pi R^2 - \pi \frac{R^2}{4} = \frac{3}{4} R^2 \pi$$

$$\boxed{x_G} = - \frac{\pi R^2}{4} \cdot \frac{4}{3 R^2 \pi} \cdot \frac{R}{2} = \boxed{-\frac{R}{6}}$$

9) Disco con foro quadrato : massa m



Disco : massa m_1, R

foro : massa m_2, $d = R$

$\Downarrow$

$l_2 = \frac{R}{\sqrt{2}}$

$G_1 \equiv O$

$G_2 = \left(\frac{R}{4}\sqrt{2}, \frac{R}{4}\sqrt{2}\right)$

$G \in$ bisettrice 1°, 3° quadrante

$A_1 = \pi R^2$

$A_2 = \frac{R^2}{2}$

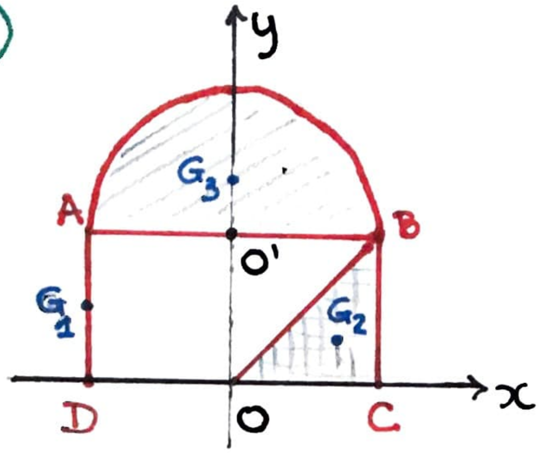
$$\boxed{x_G (= y_G)} = \frac{A_1 x_{G_1} - A_2 x_{G_2}}{A} = - \frac{R^2}{2} \cdot \frac{R}{4}\sqrt{2} \Big/ \left(\pi R^2 - \frac{R^2}{2}\right)$$

$$= \boxed{- \frac{R\sqrt{2}}{4(2\pi - 1)}}$$

$\Rightarrow G \in$ bisettrice, nel 3° quadrante.

Esercizi tipo esame

(1)



Corpo rigido costituito da:

semidisco : massa $\frac{m}{2}$, R

asta AD : massa m , $l = R$

triangolo rettangolo isoscele:

m , $\overline{OC} = \overline{BC} = R$.

Baricentro di AD : $G_1 = \left(-R, \frac{R}{2}\right)$

Baricentro di $O\hat{C}B$: $G_2 = \left(\frac{2}{3}R, \frac{R}{3}\right)$

Baricentro del semidisco : $G_3 = \left(0, R + \frac{4R}{3\pi}\right)$

$$x_G = \frac{m_1 x_{G_1} + m_2 x_{G_2} + m_3 x_{G_3}}{m_1 + m_2 + m_3} = \frac{\frac{m}{2}\cdot(0) + m\left(\frac{2}{3}R\right) + m(-R)}{m + m + \frac{m}{2}}$$

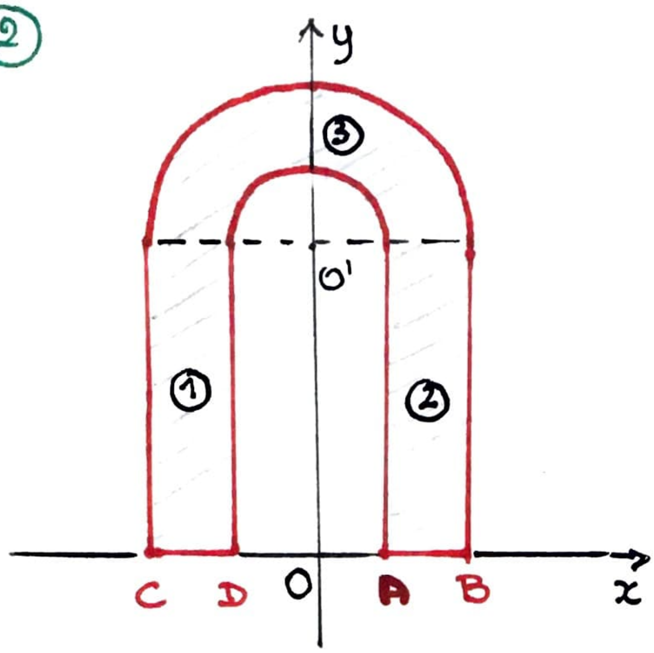
$$= \frac{-\frac{R}{3}}{\frac{5}{2}} = -\frac{2}{15}R$$

$$y_G = \frac{m\cdot\left(\frac{R}{2}\right) + m\cdot\left(\frac{R}{3}\right) + \frac{m}{2}\left(R + \frac{4R}{3\pi}\right)}{\frac{5}{2}m}$$

$$= \frac{2}{5}\left(\frac{1}{2} + \frac{1}{3} + \frac{1}{2} + \frac{2}{3\pi}\right)R = \frac{4}{15}R\left(\frac{2\pi+1}{\pi}\right)$$

$$\Rightarrow G\left(-\frac{2}{15}R ; \frac{4}{15}R\left(\frac{2\pi+1}{\pi}\right)\right)$$

②



Sistema rigido scomponibile in due rettangoli uguali ① ② e una semicorona circolare ③

$m_{TOT} = m$

$m = m_1 + m_2 + m_3$

$m_1 = m_2$

$\overline{OA} = R \quad \overline{OB} = 2R \quad \overline{OO'} = \pi R$

$A_1 = A_2 = R \cdot \pi R = \pi R^2$

$$A_3 = \underbrace{\frac{\pi \cdot 4R^2}{2}}_{A_3'} - \underbrace{\frac{\pi R^2}{2}}_{A_3''} = \frac{3}{2}\pi R^2$$

$$m = \rho\left(\pi R^2 + \pi R^2 + \frac{3}{2}\pi R^2\right) = \rho\,\frac{7}{2}\pi R^2 \rightarrow \rho = \frac{2m}{7\pi R^2}$$

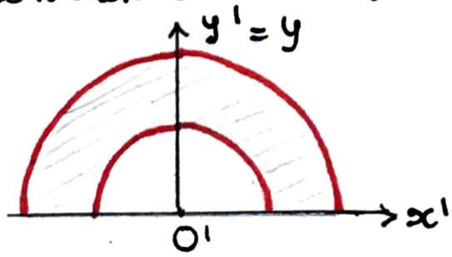
$$m_1 = m_2 = \frac{2m}{7\pi R^2} \cdot \pi R^2 = \frac{2}{7}m$$

$$m_3 = \frac{3}{7}m$$

$$G_1\left(-\frac{3}{2}R, \frac{\pi R}{2}\right); \quad G_2\left(\frac{3}{2}R, \frac{\pi R}{2}\right)$$

Per determinare G_3 devo calcolare G_3' e G_3'' rispetto ad $O'x'y'$.

Considero solo la semicorona



$$O'G_3' = \frac{2}{3}(2R)\left.\frac{\sin\alpha}{\alpha}\right|_{\alpha=\frac{\pi}{2}} = \frac{8}{3}\frac{R}{\pi}$$

$$O'G_3'' = \frac{2}{3}(R)\left.\frac{\sin\alpha}{\alpha}\right|_{\alpha=\pi/2} = \frac{4}{3}\frac{R}{\pi}$$

$$m_3 = m_3' - m_3''$$

$$m_3' = \rho A_3' = \frac{2m}{7\pi R^2} \cdot 2\pi R^2 = \frac{4}{7}m \quad , \quad m_3'' = \frac{1}{7}m$$

$$\overline{O'G_3} = \frac{\frac{4}{7}\cdot\frac{8R}{3\pi} - \frac{1}{7}\cdot\frac{4R}{3\pi}}{\frac{3}{7}} = \frac{28}{9}\frac{R}{\pi}$$

$$\Rightarrow G_3\left(0,\ \pi R + \frac{28}{9}\frac{R}{\pi}\right)$$

Pertanto il baricentro G del sistema ha coordinate:

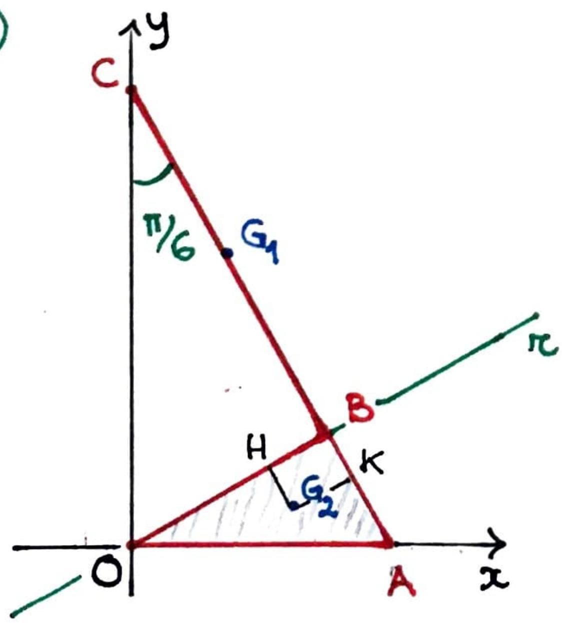
$$\boxed{x_G \equiv 0} \qquad \frac{m_1 x_{G_1} + m_2 \overset{= -x_{G_1}}{x_{G_2}} + m_3 \overset{=0}{x_{G_3}}}{\underbrace{m_1+m_2+m_3}_{m}} = 0$$

$$\boxed{y_G} = \frac{m_1 y_{G_1} + m_2 \overset{= y_{G_1}}{y_{G_2}} + m_3 y_{G_3}}{m}$$

$$= 2\cdot\frac{2}{7}\cdot\frac{\pi R}{2} + \frac{3}{7}\cdot\left(\pi R + \frac{28}{9}\frac{R}{\pi}\right)$$

$$= \frac{5}{7}\pi R + \frac{4}{3}\frac{R}{\pi} = \boxed{\frac{(15\pi^2+28)\,R}{21\pi}}$$

③



sistema rigido costituito da:

- asta BC di densità lineare ℓ
- triangolo rettangolo $O\hat{B}A$ di densità superficiale s.

Dati: $\overline{OA} = 2R$

$O\hat{C}A = \pi/6$

$\overline{OA} = 2R \Rightarrow \overline{AB} = R$ e $\overline{AC} = 4R \Rightarrow \overline{BC} = 3R$

$\Rightarrow \overline{OB} = \sqrt{3}R$ e $\overline{OC} = 2\sqrt{3}R$

$\overline{G_2H} = \frac{1}{3}\overline{AB} = \frac{R}{3}$

$\overline{G_2K} = \frac{1}{3}\overline{OB} = \frac{\sqrt{3}}{3}R$

$$\overline{CG_1} = \frac{3}{2}R$$

$$x_{G_1} = \frac{3}{2}R \sin\frac{\pi}{6} = \frac{3}{4}R$$

$$y_{G_1} = \overline{OC} - \overline{CG_1}\cos\frac{\pi}{6} = 2\sqrt{3}R - \frac{3}{2}R \cdot \frac{\sqrt{3}}{2} = \frac{5}{4}\sqrt{3}R$$

$$\Rightarrow G_1\left(\frac{3}{4}R, \frac{5}{4}\sqrt{3}R\right)$$

$$x_{G_2} = x_H + \overline{G_2H}\sin\frac{\pi}{6} = \overline{OH}\cos\frac{\pi}{6} + \overline{G_2H}\sin\frac{\pi}{6}$$

$$= \frac{2}{3}\cdot\sqrt{3}R\cdot\frac{\sqrt{3}}{2} + \frac{1}{3}R\cdot\frac{1}{2} = R + \frac{R}{6} = \frac{7}{6}R$$

$$y_{G_2} = y_H - \overline{G_2H}\cos\frac{\pi}{6} = \overline{OH}\sin\frac{\pi}{6} - \overline{G_2H}\cos\frac{\pi}{6}$$

$$= \frac{2}{3}\cdot\sqrt{3}R\cdot\frac{1}{2} - \frac{R}{3}\cdot\frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{6}R$$

$$\Rightarrow G_2\left(\frac{7}{6}R, \frac{\sqrt{3}}{6}R\right)$$

$$m_1 = \ell\,\overline{BC} = \ell\cdot 3R$$

$$m_2 = s\cdot\frac{\overline{OB}\cdot\overline{BA}}{2} = s\cdot\frac{\sqrt{3}R\cdot R}{2} = s\frac{\sqrt{3}}{2}R^2$$

Si chiede di determinare il legame tra "ℓ" ed "s" affinchè il baricentro "G" del sistema appartenga alla retta r passante per O e B.

La retta ha equazione $y = \operatorname{tg}\pi/6\, x = \frac{\sqrt{3}}{3}x$

$$\Rightarrow \boxed{y_G = \frac{\sqrt{3}}{3}x_G}$$

$$\begin{cases} x_G = \dfrac{m_1 x_{G1} + m_2 x_{G2}}{m_1 + m_2} \\ y_G = \dfrac{m_1 y_{G1} + m_2 y_{G2}}{m_1 + m_2} \end{cases} \Rightarrow \underline{\sqrt{3}\left(m_1 y_{G_1} + m_2 y_{G2}\right) = m_1 x_{G_1} + m_2 x_{G_2}}$$

Quindi

$$m_1\left(\sqrt{3}\, y_{G_1} - x_{G_1}\right) = m_2\left(x_{G_2} - \sqrt{3}\, y_{G_2}\right)$$

$$\ell \cdot 3R\left(\sqrt{3} \cdot \frac{5\sqrt{3}}{4}R - \frac{3}{4}R\right) = s\,\frac{\sqrt{3}}{2}R^2\left(\frac{7}{6}R - \sqrt{3} \cdot \frac{\sqrt{3}}{6}R\right)$$

$$9\ell R^2 = \frac{\sqrt{3}}{3} s R^3 \quad \Rightarrow \quad \boxed{\ell = \frac{\sqrt{3}\, R s}{27}}$$

Come conseguenza:

$$m_1 = \ell \cdot 3R = \frac{\sqrt{3}}{9} R^2 s$$

$$m_2 = \frac{\sqrt{3}}{2} R^2 s$$

$m_1 + m_2 = \underline{m}$ massa totale

$$\Rightarrow m = \frac{\sqrt{3}}{9} R^2 s + \frac{\sqrt{3}}{2} R^2 s = \frac{11}{18}\sqrt{3} R^2 s$$

quindi $s = \dfrac{18}{11\sqrt{3} R^2} m = \dfrac{6\sqrt{3}}{11 R^2} m$

$$\ell = \frac{\sqrt{3}}{27} R \cdot \frac{6\sqrt{3}}{11 R^2} m = \frac{2}{33R} m$$

$$\Rightarrow \begin{cases} m_1 = \dfrac{\sqrt{3}}{9} R^2 \cdot \dfrac{6\sqrt{3}}{11 R^2} m = \dfrac{2}{11} m \\ m_2 = \dfrac{9}{11} m \end{cases}$$

FIGURE NON OMOGENEE

1) asta AB di massa m e lunghezza ℓ tale che la densità di massa varia con la legge

$$\boxed{\rho(P) = k\,|P-A|} \quad , \; k>0$$

lineare

$\overline{PA} = s \Rightarrow \rho(s) = ks$

$$m = \int_0^\ell \rho(s)\,ds = \int_0^\ell ks\,ds = k \cdot \frac{\ell^2}{2} \Rightarrow k = \frac{2m}{\ell^2}$$

$$x_G = \frac{1}{m}\int_0^\ell \rho(s)\,s\,ds = \frac{1}{m}\int_0^\ell ks^2\,ds = \frac{2}{\ell^2}\cdot\frac{1}{3}\ell^3 = \boxed{\frac{2\ell}{3}}$$

$\shortparallel$

$\boxed{s_G}$

$$\Rightarrow \boxed{\overline{AG} = \frac{2}{3}\overline{AB}}$$

2) asta AB di massa m e lunghezza ℓ la cui densità di massa varia con la legge: $\boxed{\rho(P) = k\,|P-A|^2}$, $k>0$

quadratica

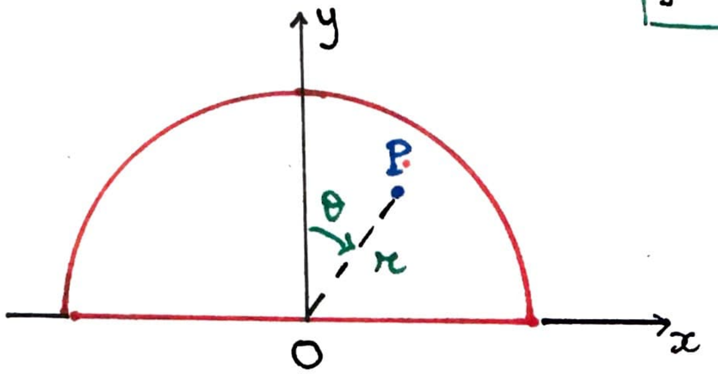
$\overline{PA} = s \Rightarrow \rho(s) = ks^2$

$$m = \int_0^\ell \rho(s)\,ds = \int_0^\ell ks^2\,ds = k\cdot\frac{\ell^3}{3} \Rightarrow k = \frac{3m}{\ell^3}$$

$$\boxed{s_G} (= x_G) = \frac{1}{m}\int_0^\ell \rho(s)\,s\,ds = \frac{1}{m}\int_0^\ell ks^3\,ds = \frac{3}{\ell^3}\cdot\frac{1}{4}\ell^4 = \boxed{\frac{3\ell}{4}}$$

$$\Rightarrow \boxed{\overline{AG} = \frac{3}{4}\overline{AB}}$$

3) semidisco di massa m e raggio R la cui densità varia con la legge: $\boxed{\rho(P) = k\left(1 + \frac{1}{R}\overline{OP}\right)}$, $k > 0$



$$\Downarrow$$

$\rho(r) = k\left(1 + \frac{r}{R}\right)$ radiale

$G \in Oy$ per simmetria

$$m = \int_{-\pi/2}^{\pi/2}\int_0^R k\left(1+\frac{r}{R}\right) r\,dr\,d\theta = k\int_{-\pi/2}^{\pi/2} d\theta \int_0^R \left(r + \frac{r^2}{R}\right) dr$$

$$= k \cdot \pi \cdot \left(\frac{1}{2}R^2 + \frac{1}{3}R^2\right) = \frac{5}{6}k\pi R^2 \Rightarrow k = \frac{6m}{5\pi R^2}$$

$$y_G = \frac{1}{m}\int_{-\pi/2}^{\pi/2}\int_0^R k\left(1+\frac{r}{R}\right) r\cos\theta\, r\,dr\,d\theta$$

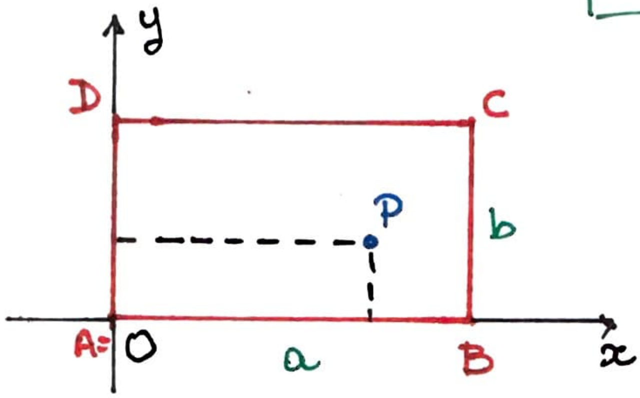
$$= \frac{6}{5\pi R^2}\int_{-\pi/2}^{\pi/2}\cos\theta\,d\theta \int_0^R \left(r^2 + \frac{r^3}{R}\right)dr$$

$$= \frac{6}{5\pi R^2}\cdot \underbrace{(\operatorname{sen}\theta)\Big|_{-\pi/2}^{\pi/2}}_{=2} \cdot \left(\frac{1}{3}R^3 + \frac{1}{4}R^3\right) = \frac{12}{5\pi R^2}\cdot\frac{7}{12}R^3$$

$$= \frac{7}{5}\frac{R}{\pi}$$

se non ci si accorge della simmetria radiale, dal calcolo diretto $x_G \equiv 0$.

4) rettangolo di massa m e lati a, b la cui densità varia con la legge $\boxed{\rho(P) = k\,\overline{OP}^2}$, $k>0$



$\overline{OP}^2 = x^2 + y^2$

$P(x, y)$

$$m = \int_0^a \int_0^b k(x^2+y^2)\,dx\,dy = k\left[\int_0^a\int_0^b x^2\,dx\,dy + \int_0^a\int_0^b y^2\,dx\,dy\right]$$

$$= k\left[\int_0^a x^2\,dx \int_0^b dy + \int_0^a dx \int_0^b y^2\,dy\right]$$

$$= k\left(\frac{1}{3}a^3 \cdot b + a \cdot \frac{1}{3} b^3\right) = k\,\frac{ab}{3}(a^2+b^2)$$

$$\boxed{x_G} = \frac{1}{m}\int_0^a\int_0^b k(x^2+y^2)\,x\,dx\,dy$$

$$= \frac{3}{ab(a^2+b^2)}\left[\int_0^a\int_0^b x^3\,dx\,dy + \int_0^a\int_0^b x y^2\,dx\,dy\right]$$

$$= \frac{3}{ab(a^2+b^2)}\left[\frac{1}{4}a^4 \cdot b + \frac{1}{2}a^2 \cdot \frac{1}{3}b^3\right]$$

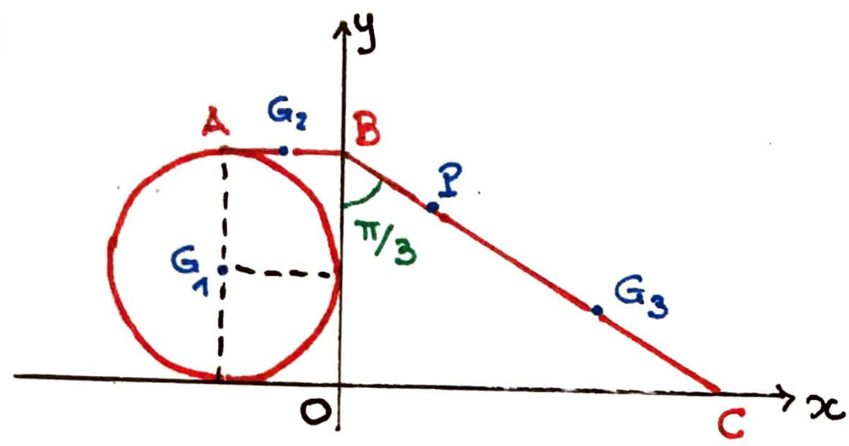
$$= \frac{3}{ab(a^2+b^2)} \cdot \frac{a^2 b(3a^2+2b^2)}{12} = \boxed{\frac{a(3a^2+2b^2)}{4\,(a^2+b^2)}}$$

Analogamente

$$\boxed{y_G = \frac{b}{4}\,\frac{(2a^2+3b^2)}{(a^2+b^2)}}$$

Esercizio tipo esame

(1)



Dati $\overline{AB} = R$

$O\hat{B}C = \pi/3$

Sistema rigido costituito da:

1) disco omogeneo, di massa m, raggio R.

2) asta omogenea AB, di massa m e lunghezza R

3) asta NON OMOGENEA BC, di massa m la cui densità varia con la legge $\rho(P) = k|P-B|$, $k>0$.

lineare

Disco $\Rightarrow$ $G_1(-R, R)$

Asta $\overline{AB}$ $\Rightarrow$ $G_2\left(-\frac{R}{2}, 2R\right)$

Asta $\overline{BC}$: $\overline{BC} = 2\,\overline{OB} = 4R$

Poiché $\rho(s) = ks$ $\Rightarrow$ $\overline{BG_3} = \frac{2}{3}\overline{BC}$ $\Rightarrow$ $\overline{BG_3} = \frac{8}{3}R$

quindi $\overline{G_3C} = \frac{4}{3}R$

$$\begin{cases} x_{G_3} = \frac{8}{3}R\sin\frac{\pi}{3} = \frac{4}{3}\sqrt{3}R \\ y_{G_3} = 2R - \frac{8}{3}R\cos\frac{\pi}{3} = 2R - \frac{4}{3}R = \frac{2}{3}R \end{cases} \Rightarrow G_3\left(\frac{4}{3}\sqrt{3}R, \frac{2}{3}R\right)$$

$$x_G = \frac{m_1 x_{G_1} + m_2 x_{G_2} + m_3 x_{G_3}}{m_1 + m_2 + m_3} = \frac{-R - \frac{R}{2} + \frac{4}{3}\sqrt{3}R}{3}$$

$$= \frac{(8\sqrt{3} - 9)R}{18}$$

$$y_G = \frac{m_1 y_{G_1} + m_2 y_{G_2} + m_3 y_{G_3}}{m_1 + m_2 + m_3} = \frac{R + 2R + \frac{2}{3}R}{3} = \frac{11}{9}R$$

(2) stessa figura, ma: • disco omogeneo
• asta BC omogenea
• asta AB non omogenea con

$\rho(P) = k\,|P - B|$, $k > 0$

DISCO $\Rightarrow G_1(-R, R)$

Asta $\overline{BC} \Rightarrow G_3(\sqrt{3}R, R)$

Asta $\overline{AB}$: G_2 ; $\overline{BG_2} = \frac{2}{3}\overline{AB} \Rightarrow G_2\left(-\frac{2}{3}R, 2R\right)$

$$\begin{cases} x_G = \dfrac{m(-R) + m\left(-\frac{2}{3}R\right) + m(\sqrt{3}R)}{3m} = \dfrac{(3\sqrt{3}-5)R}{9} \\[2ex] y_G = \dfrac{m(R) + m(2R) + m(R)}{3m} = \dfrac{4}{3}R \end{cases}$$