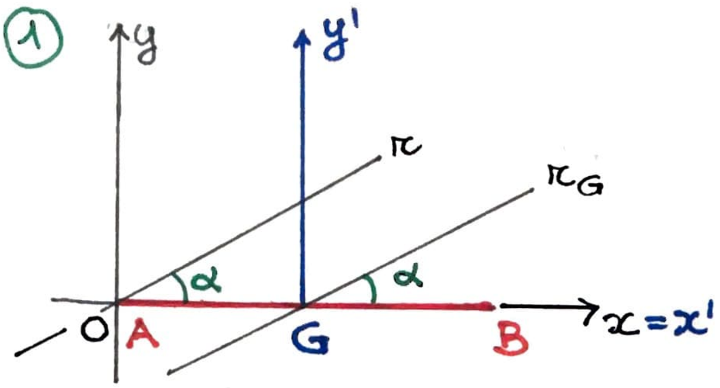
MATRICI D'INERZIA

- ASTE OMOGENEE E NON
- RETTANGOLI
- TRIANGOLI
- CIRCONFERENZA e SUE PARTI
- DISCO E SUE PARTI
- LAMINE CON FORI
- ESERCIZI TIPO ESAME

ASTA OMOGENEA

(1)



$\overline{AB}: m, L$

$\overline{AB} \in Ox$

$I_{11} = 0 \quad ; \quad I_{12} = I_{13} = I_{23} = 0$

$I_{22} = I_{33}$

$$I_{22} = \int_0^L \rho x^2 dx = \frac{m}{L} \frac{1}{3} L^3 = \frac{mL^2}{3}$$

$$\Rightarrow \underset{\sim}{I}_O = \frac{mL^2}{3} \operatorname{diag}(0,1,1)$$

$$I_{11}^G = 0, \quad I_{12}^G = I_{13}^G = I_{23}^G = 0, \quad I_{22}^G = I_{33}^G$$

$$I_{22}^G = \int_{-L/2}^{L/2} \rho x'^2 dx' = \frac{m}{L} \cdot \frac{1}{3} \cdot \left[\left(\frac{L}{2}\right)^3 - \left(-\frac{L}{2}\right)^3\right] = \frac{1}{12} mL^2$$

$$\Rightarrow \underset{\sim}{I}_G = \frac{mL^2}{12} \operatorname{diag}(0,1,1)$$

vale th. di Huygens : $I_{22} = I_{22}^G + md^2 \qquad d = \frac{L}{2}$

$$\frac{mL^2}{3} = \frac{mL^2}{12} + \frac{mL^2}{4} \quad \text{ok!}$$

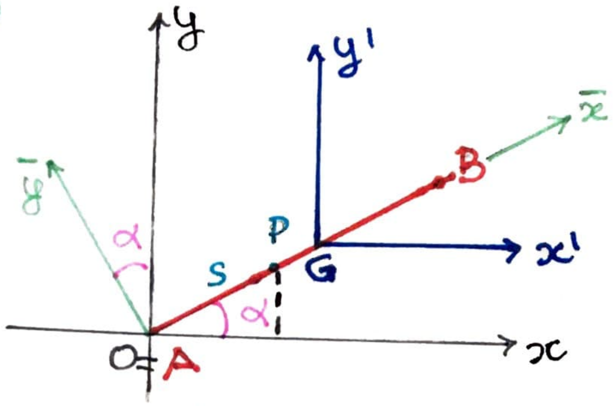
$$I_r = \underset{\sim}{I}_O \bar{r} \cdot \bar{r} \qquad \bar{r} = (\cos\alpha, \sin\alpha, 0)$$

$$= \frac{mL^2}{3} \begin{bmatrix} 0 & & \\ & 1 & \\ & & 1 \end{bmatrix} \begin{bmatrix} \cos\alpha \\ \sin\alpha \\ 0 \end{bmatrix} \cdot \begin{bmatrix} \cos\alpha \\ \sin\alpha \\ 0 \end{bmatrix} = \frac{mL^2}{3} \sin^2\alpha$$

analogamente

$$I_r^G = \underset{\sim}{I}_G \bar{r}_G \cdot \bar{r}_G = \frac{mL^2}{12} \sin^2\alpha$$

②



$$\underset{\sim}{I}_O = \begin{bmatrix} I_{11} & +I_{12} & 0 \\ I_{12} & I_{22} & 0 \\ 0 & 0 & I_{11}+I_{22} \end{bmatrix}$$

dove:

$$I_{11} = \int_0^L \rho y^2 \, ds = \frac{m}{L}\int_0^L s^2 \sin^2\alpha \, ds = \frac{m}{L}\sin^2\alpha \cdot \frac{1}{3}L^3$$

$$= \frac{mL^2}{3}\sin^2\alpha$$

$$I_{33} = \frac{mL^2}{3} \quad \text{(vedi caso precedente)}$$

$$\Rightarrow I_{22} = \frac{mL^2}{3}(1-\sin^2\alpha) = \frac{mL^2}{3}\cos^2\alpha$$

oppure direttamente

$$I_{22} = \int_0^L \rho x^2 \, ds = \frac{m}{L}\int_0^L s^2\cos^2\alpha \, ds = \frac{mL^2}{3}\cos^2\alpha$$

$$I_{12} = -\int_0^L \rho xy \, ds = -\frac{m}{L}\int_0^L s^2 \sin\alpha\cos\alpha \, ds =$$

$$= -\frac{mL^2}{3}\sin\alpha\cos\alpha$$

- Nel riferimento $O\bar{x}\bar{y}$ la matrice $\underset{\sim}{\bar{I}}_O$ è diagonale

$$\underset{\sim}{\bar{I}}_O = \frac{mL^2}{3}\,\mathrm{diag}(0,1,1) \quad \text{(vedi caso precedente)}$$

$$\Rightarrow I_{11} = \underset{\sim}{\bar{I}}_O\, \bar{\imath}\cdot\bar{\imath} \qquad \bar{\imath} = \cos\alpha\, \bar{J}_1 - \sin\alpha\, \bar{J}_2$$

$$= \frac{mL^2}{3}\sin^2\alpha$$

$$I_{22} = \underset{\sim}{\bar{I}}_O\, \bar{J}\cdot\bar{J} \qquad \bar{J} = \sin\alpha\, \bar{J}_1 + \cos\alpha\, \bar{J}_2$$

$$= \frac{mL^2}{3}\cos^2\alpha$$

$$I_{11}^{G} = \int_{-L/2}^{L/2} \rho y^2 \, ds = \frac{m}{L} \cdot \sin^2\alpha \cdot \frac{1}{3} \cdot 2 \cdot \frac{L^3}{8} = \frac{mL^2}{12} \sin^2\alpha$$

$$I_{22}^{G} = \int_{-L/2}^{L/2} \rho x^2 \, ds = \frac{mL^2}{12} \cos^2\alpha$$

ma anche applicando Huygens:

$$I_{11}^{G} = I_{11} - md^2 \qquad d = y_G$$

$$= \frac{mL^2}{3} \sin^2\alpha - m \cdot \left(\frac{L}{2} \sin\alpha\right)^2 = \frac{mL^2}{12} \sin^2\alpha$$

$$I_{22}^{G} = I_{22} - md'^2 \qquad d' = x_G$$

$$= \frac{mL^2}{3} \cos^2\alpha - m \left(\frac{L}{2} \cos\alpha\right)^2 = \frac{mL^2}{12} \cos^2\alpha$$

$$I_{12}^{G} = -\int_{-L/2}^{L/2} \rho xy \, ds = -\frac{m}{L} \sin\alpha\cos\alpha \int_{-L/2}^{L/2} s^2 \, ds$$

$$= -\frac{mL^2}{12} \sin\alpha\cos\alpha$$

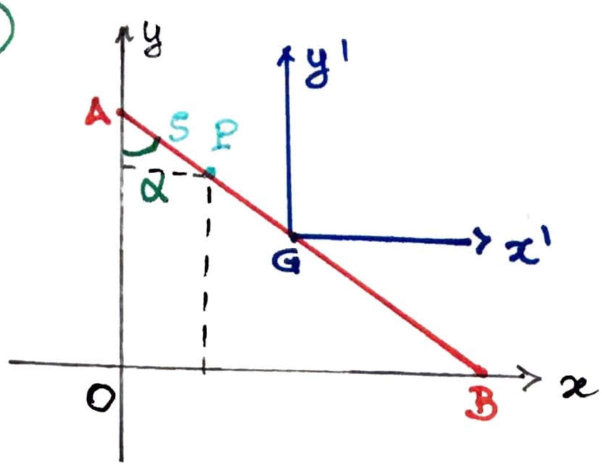
ma anche:

$$I_{12} = I_{12}^{G} - m x_G y_G$$

$$-\frac{mL^2}{3} \sin\alpha\cos\alpha = -\frac{mL^2}{12} \sin\alpha\cos\alpha - m \frac{L}{2} \sin\alpha \cdot \frac{L}{2} \cos\alpha \quad \text{ok!}$$

$$\Rightarrow \underset{\sim}{I}_G = \begin{bmatrix} \frac{mL^2}{12} \sin^2\alpha & -\frac{mL^2}{12} \sin\alpha\cos\alpha & 0 \\ -\frac{mL^2}{12} \sin\alpha\cos\alpha & \frac{mL^2}{12} \cos^2\alpha & 0 \\ 0 & 0 & \frac{mL^2}{12} \end{bmatrix}$$

(3)



$$I_{22} = \int_0^L \rho x^2 ds$$

$$= \frac{m}{L} \int_0^L s^2 \sin^2\alpha \, ds$$

$$= \frac{mL^2}{3} \sin^2\alpha$$ (già calcolato in (1))

$$I_{11} = \int_0^L \rho y^2 ds = \int_0^L \rho (L-s)^2 \cos^2\alpha \, ds$$

$$= \frac{m}{L} \cos^2\alpha \int_0^L (L^2 - 2Ls + s^2) ds$$

$$= \frac{m}{L} \cos^2\alpha \left(L^2 s - 2L \cdot \frac{1}{2} s^2 + \frac{1}{3} s^3 \right)_0^L$$

$$= \frac{1}{3} mL^2 \cos^2\alpha$$ (già calcolato in (1))

$$I_{33} = \frac{mL^2}{3}$$

$$I_{12} = -\int_0^L \rho \, xy \, ds = -\frac{m}{L} \int_0^L (s \sin\alpha)(L-s)\cos\alpha \, ds$$

$$= -\frac{m}{L} \sin\alpha \cos\alpha \int_0^L (Ls - s^2) ds$$

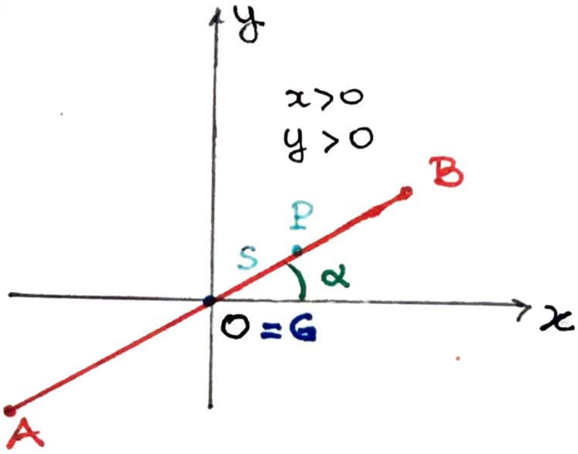
$$= -\frac{m}{L} \sin\alpha \cos\alpha \left(L \cdot \frac{1}{2} s^2 - \frac{1}{3} s^3 \right)_0^L$$

$$= -\frac{1}{6} mL^2 \sin\alpha \cos\alpha$$

$$\underset{\sim}{I}_O = \begin{bmatrix} \frac{1}{3} mL^2 \cos^2\alpha & -\frac{1}{6} mL^2 \sin\alpha\cos\alpha & 0 \\ -\frac{1}{6} mL^2 \sin\alpha\cos\alpha & \frac{1}{3} mL^2 \sin^2\alpha & 0 \\ 0 & 0 & \frac{mL^2}{3} \end{bmatrix}$$

- calcolare $\underset{\sim}{I}_G$

4



$$I_{11} = \frac{1}{12} mL^2 \sin^2\alpha \quad (\text{visto in } 1)$$

$$I_{22} = \frac{1}{12} mL^2 \cos^2\alpha \quad (\quad '' \quad)$$

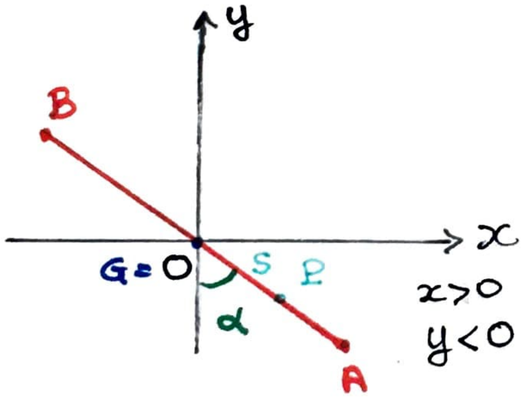
$$I_{33} = \frac{1}{12} mL^2$$

$$I_{12} = -\int_{-L/2}^{L/2} \rho\, xy\, ds = -\frac{m}{L}\int_{-L/2}^{L/2} s\cos\alpha\, s\sin\alpha\, ds$$

$$= -\frac{m}{L}\sin\alpha\cos\alpha \int_{-L/2}^{L/2} s^2 ds = -\frac{m}{L}\sin\alpha\cos\alpha\, \frac{1}{3}\cdot 2\cdot \frac{L^3}{8}$$

$$= -\frac{1}{12} mL^2 \sin\alpha\cos\alpha$$

se invece



$$I_{11} = \frac{mL^2}{12}\cos^2\alpha$$

$$I_{22} = \frac{mL^2}{12}\sin^2\alpha$$

$$I_{33} = \frac{mL^2}{12}$$

$$I_{12} = -\int_{-L/2}^{L/2} \rho\, xy\, ds = -\frac{m}{L}\int_{-L/2}^{L/2} (s\sin\alpha)(-s\cos\alpha)\, ds$$

$$= +\frac{m}{L}\sin\alpha\cos\alpha \int_{-L/2}^{L/2} s^2 ds = +\frac{1}{12} mL^2 \sin\alpha\cos\alpha$$

ASTA NON OMOGENEA

(1) $\rho(P) = k|P-O|, \ k>0 \implies \rho(x) = kx$

$$m = \int_0^L kx\,dx = \frac{kL^2}{2}$$

$$I_{11} = 0$$

$$I_{22} = I_{33} = \int_0^L kx\,x^2\,dx = \frac{2m}{L^2}\cdot\frac{1}{4}L^4 = \frac{mL^2}{2}$$

$$\underset{\sim}{I}_O = \frac{mL^2}{2}\,\mathrm{diag}(0,1,1)$$

$$x_G = \frac{2}{3}L$$

$$I_{22}^G = I_{22} - md^2 \qquad d = x_G$$

$$= \frac{mL^2}{2} - m\cdot\frac{4}{9}L^2 = \frac{1}{18}mL^2$$

$$\underset{\sim}{I}_G = \frac{mL^2}{18}\,\mathrm{diag}(0,1,1)$$

(2) $\rho(s) = ks, \ k>0 \qquad \overline{AP} = s$

$$I_{11} = \int_0^L ks\,(s\sin\alpha)^2\,ds = \frac{2m}{L^2}\,\frac{1}{4}L^4\sin^2\alpha = \frac{mL^2}{2}\sin^2\alpha$$

$$I_{22} = I_{33} - I_{11} = \frac{mL^2}{2}\cos^2\alpha$$

$$I_{12} = -\int_0^L ks\,(s\sin\alpha)(s\cos\alpha)\,ds = -\frac{2m}{L^2}\sin\alpha\cos\alpha\,\frac{1}{4}L^4$$

$$= -\frac{mL^2}{2}\sin\alpha\cos\alpha$$

3 $\rho(s) = ks$, $k>0$, $\overline{AP} = s$

$$I_{11} = \int_0^L ks\,(L-s)^2 \cos^2\alpha \, ds = \frac{2m}{L^2} \cos^2\alpha \int_0^L (L^2 s - 2Ls^2 + s^3)\, ds$$

$$= \frac{2m}{L^2} \cos^2\alpha \left(L^2 \cdot \frac{1}{2} L^2 - 2L \cdot \frac{1}{3} L^3 + \frac{1}{4} L^4 \right)$$

$$= 2m \frac{L^2}{12} \cos^2\alpha = \frac{mL^2}{6} \cos^2\alpha$$

$$I_{22} = \int_0^L ks\; s^2 \sin^2\alpha \, ds = \frac{2m}{L^2} \sin^2\alpha \; \frac{1}{4} L^4 = \frac{mL^2}{2} \sin^2\alpha$$

$$I_{33} = \frac{mL^2}{6} \left(1 + 2 \sin^2\alpha \right)$$

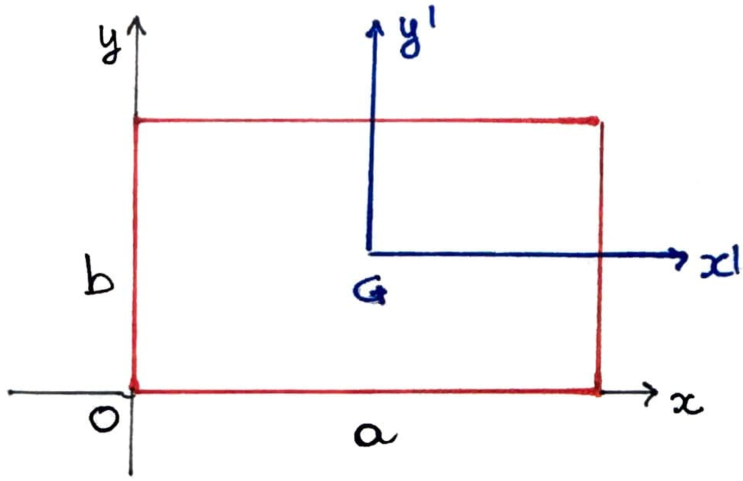
$$I_{12} = - \int_0^L ks\,(\ell - s) \cos\alpha \; s \sin\alpha \, ds$$

$$= - \frac{2m}{L^2} \sin\alpha \cos\alpha \int_0^L s^2 (\ell - s)\, ds$$

$$= - \frac{2m}{L^2} \sin\alpha \cos\alpha \; \frac{1}{12} L^4 = - \frac{mL^2}{6} \sin\alpha \cos\alpha$$

RETTANGOLO

Omogeneo : massa m e lati a, b



Calcolare :

- $\underset{\sim}{I}_O$
- $\underset{\sim}{I}_G$

$$I_{11} = \int_0^a \int_0^b \rho\, y^2\, dx\, dy = \frac{m}{ab} \int_0^a dx \int_0^b y^2\, dy = \frac{m}{ab} \cdot a\, \frac{1}{3} b^3$$

$$= \frac{m b^2}{3}$$

$$I_{22} = \int_0^a \int_0^b \rho\, x^2\, dx\, dy = \frac{m a^2}{3}$$

$$I_{33} = \frac{m}{3}(a^2 + b^2)$$

$$I_{12} = -\int_0^a \int_0^b \rho\, xy\, dx\, dy = -\frac{m}{ab} \cdot \frac{1}{2} a^2 \cdot \frac{1}{2} b^2 = -\frac{m a b}{4}$$

Per determinare $\underset{\sim}{I}_G$ o si procede con la definizione (come per $\underset{\sim}{I}_O$) o si applica il th. di Huygens.
Il rif. $Gx'y'z'$ è principale d'inerzia. $\Rightarrow I_{12}^G \equiv 0$.

$$I_{11}^G = I_{11} - m \underset{y_G^2}{\underset{\parallel}{d^2}} = \frac{m b^2}{3} - m \frac{b^2}{4} = \frac{m b^2}{12}$$

$$I_{22}^G = I_{22} - m x_G^2 = \frac{m a^2}{12}$$

$$I_{33}^G = \frac{m}{12}(a^2 + b^2)$$

Per il calcolo di I_{12} si può anche ricordare che:

$$I_{12} = \underset{\underset{0}{\parallel}}{I_{12}^{G}} - m x_G y_G = -m \frac{a}{2} \cdot \frac{b}{2} = -\frac{mab}{4}$$

$$\underset{\sim}{I}_O = \begin{bmatrix} \frac{mb^2}{3} & -\frac{mab}{4} & 0 \\ -\frac{mab}{4} & \frac{ma^2}{3} & 0 \\ 0 & 0 & \frac{m}{3}(a^2+b^2) \end{bmatrix}$$

$$\underset{\sim}{I}_G = \frac{m}{12} \operatorname{diag}(b^2, a^2, a^2+b^2)$$

QUADRATO

Caso particolare del rettangolo in cui $a=b=L$.

$$\underset{\sim}{I}_O = mL^2 \begin{bmatrix} 1/3 & -1/4 & 0 \\ -1/4 & 1/3 & 0 \\ 0 & 0 & 2/3 \end{bmatrix}$$

$$\underset{\sim}{I}_G = \frac{mL^2}{12} \operatorname{diag}(1, 1, 2).$$