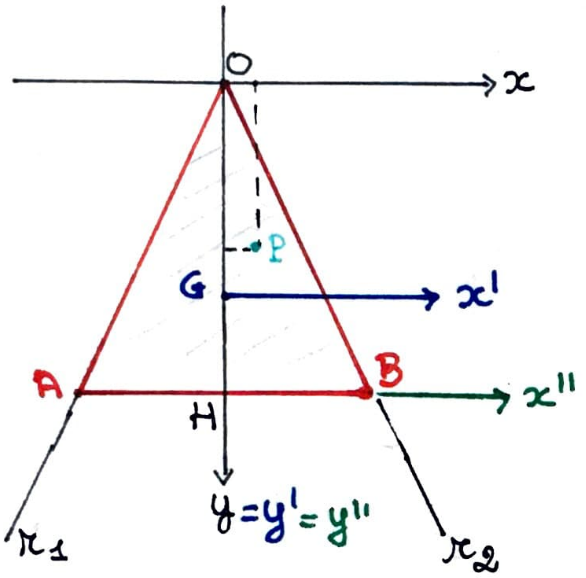
TRIANGOLO ISOSCELE

omogeneo di massa m, base a, altezza h



Oy è asse di simmetria

$Oxyz$ è principale d'inerzia

$Gx'y'z'$ " " "

$\underset{\sim}{I}_O$, $\underset{\sim}{I}_G$ diagonali

$\overline{OG} = \frac{2}{3}h$

$A\left(-\frac{a}{2}, h\right)$; $B\left(\frac{a}{2}, h\right)$

retta r_1 : $y = -\frac{2h}{a}x$

retta r_2 : $y = \frac{2h}{a}x$

$$I_{11} = \int_F \rho\, y^2\, dF = \frac{2m}{ah}\int_0^h y^2 \int_{-\frac{a}{2h}y}^{\frac{a}{2h}y} dx\, dy$$

$$= \frac{2m}{ah}\int_0^h y^2 \cdot \frac{a}{h} y\, dy = \frac{2m}{h^2}\cdot\frac{1}{4}h^4 = \frac{mh^2}{2}$$

$$I_{22} = \int_F \rho\, x^2\, dF = \frac{2m}{ah}\int_0^h \int_{-\frac{a}{2h}y}^{\frac{a}{2h}y} x^2\, dx\, dy$$

$$= \frac{2m}{ah}\cdot\int_0^h \frac{a^3}{8h^3}\cdot\frac{1}{3}y^3\cdot 2\, dy = \frac{ma^2}{6h^4}\cdot\frac{1}{4}h^4 = \frac{ma^2}{24}$$

$$I_{33} = \frac{mh^2}{2} + \frac{ma^2}{24}$$

$$I_{11}^{G} = I_{11} - m\, y_G^2 = \frac{mh^2}{2} - m \cdot \frac{4}{9} h^2 = \frac{mh^2}{18}$$

$$I_{22}^{G} \equiv I_{22} = \frac{ma^2}{24}$$

$$I_{33}^{G} = \frac{mh^2}{18} + \frac{ma^2}{24}$$

$$I_{AB} = I_{11}^{G} + m\, \overline{GH}^2 = \frac{mh^2}{18} + m\frac{h^2}{9} = \frac{mh^2}{6}$$

Osservazione

Il momento d'inerzia calcolato rispetto ad un lato vale SEMPRE $\frac{1}{6} m h^2$ dove h è l'altezza relativa a quel lato.

$$I_{33}^{H} = I_{33}^{G} + m\, \overline{GH}^2 = \frac{mh^2}{18} + \frac{ma^2}{24} + m\frac{h^2}{9} = \frac{mh^2}{6} + \frac{ma^2}{24}$$

Quindi nel rif. $Hx''y''$ che è principale d'inerzia $\underset{\sim}{I}_H$ è diagonale e vale:

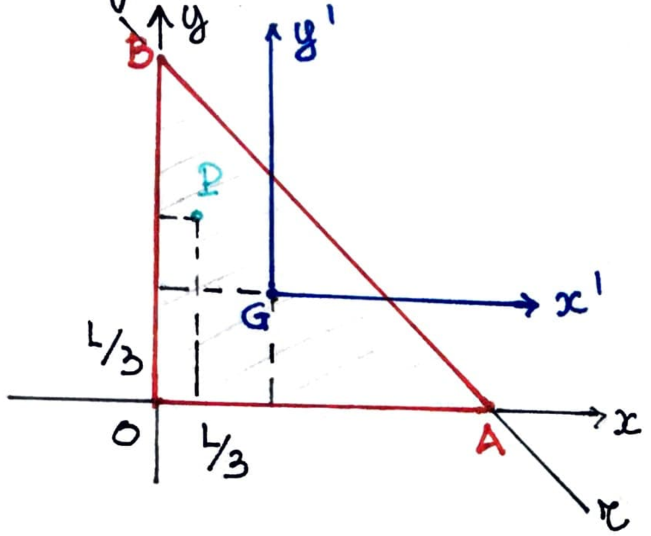
$$I_{11}^{H} \equiv I_{AB} = \frac{mh^2}{6}$$

$$I_{22}^{H} \equiv I_{22} = \frac{ma^2}{24}$$

$$I_{33}^{H} = \frac{mh^2}{6} + \frac{ma^2}{24}$$

TRIANGOLO RETTANGOLO ISOSCELE

omogeneo di massa m e cateto L.



$Oxyz$ non è principale d'inerzia

$Gx'y'z'$ non è principale d'inerzia

$G\left(\frac{L}{3}, \frac{L}{3}\right)$

$r: \ y = L - x$

$$I_{11} = I_{22} = \frac{mL^2}{6} \quad \text{(visto nel caso precedente)}$$

$$I_{33} = \frac{mL^2}{3}$$

$$I_{12} = -\int_F \rho xy \, dF = -\frac{2m}{L^2} \int_0^L \int_0^{L-x} xy \, dx \, dy$$

$$= -\frac{2m}{L^2} \int_0^L x \left(\int_0^{L-x} y \, dy \right) dx$$

$$= -\frac{2m}{L^2} \int_0^L x \cdot \frac{1}{2}(L-x)^2 \, dx = -\frac{m}{L^2} \int_0^L (L^2 x - 2Lx^2 + x^3) dx$$

$$= -\frac{m}{L^2} \left(L^2 \cdot \frac{1}{2} L^2 - 2L \cdot \frac{1}{3} L^3 + \frac{1}{4} L^4 \right) = -\frac{mL^2}{12}$$

$$\underset{\sim}{I}_O = \begin{bmatrix} \frac{mL^2}{6} & -\frac{mL^2}{12} & 0 \\ -\frac{mL^2}{12} & \frac{mL^2}{6} & 0 \\ 0 & 0 & \frac{mL^2}{3} \end{bmatrix}$$

$$I_{11}^G = I_{11} - m y_G^2 = \frac{mL^2}{6} - m\frac{L^2}{9} = \frac{mL^2}{18}$$

$$I_{22}^G = I_{11}^G = \frac{mL^2}{18}$$

$$I_{33}^G = \frac{mL^2}{9}$$

Per calcolare I_{12}^G ricordo:

$$I_{12} = I_{12}^G - m x_G y_G$$

$$I_{12}^G = I_{12} + m x_G y_G = -\frac{mL^2}{12} + \frac{mL^2}{9} = +\frac{mL^2}{36}$$

$$\underset{\sim}{I}_G = \begin{bmatrix} \frac{mL^2}{18} & \frac{mL^2}{36} & 0 \\ \frac{mL^2}{36} & \frac{mL^2}{18} & 0 \\ 0 & 0 & \frac{mL^2}{9} \end{bmatrix}$$

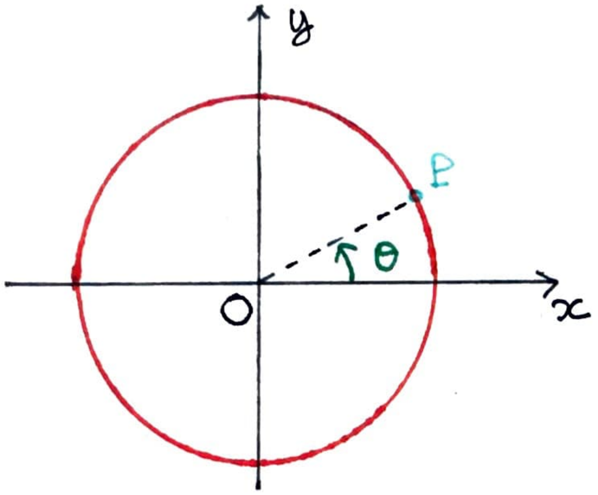
Osservazione

Il triangolo isoscele può essere visto come la somme di due triangoli rettangoli di cateti $\frac{a}{2}$, h e massa $\frac{m}{2}$.

$$I_{22} = 2\, I_{22}^* \quad \text{dove} \quad I_{22}^* = \frac{1}{6}\cdot\left(\frac{m}{2}\right)\cdot\frac{a^2}{4} = \frac{ma^2}{48}$$

$$= \frac{ma^2}{24}$$

CIRCONFERENZA



omogenea di massa m e raggio R.

$Oxyz$ principale d'inerzia

$\underset{\sim}{I}_O$ diagonale

$I_{11} = I_{22}$ per simmetria

$I_{11} + I_{22} = I_{33} \Rightarrow I_{11} = I_{22} = \frac{I_{33}}{2}$

$$\underline{I_{33}} = \int_F \rho R^2 R\,d\theta = \frac{m}{2\pi R}\int_0^{2\pi} R^3\,d\theta = \underline{mR^2}$$

OSSERVAZIONE

- $I_{33} = mR^2$ qualunque sia l'arco di circonferenza

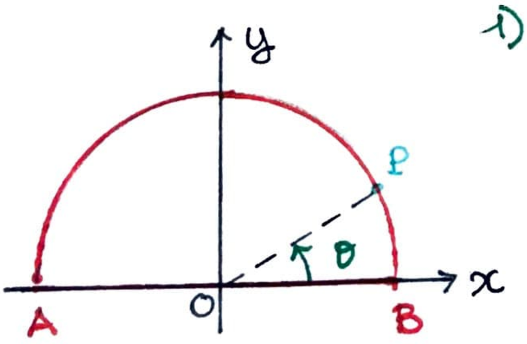
Infatti:

$$I_{33} = \int_0^{\alpha} \rho R^2 R\,d\theta = \frac{m}{R\alpha} \cdot R^3 \int_0^{\alpha} d\theta = mR^2 \quad \forall \alpha \text{ angolo.}$$

$$\Rightarrow \underset{\sim}{I}_O = mR^2 \operatorname{diag}\left(\frac{1}{2}, \frac{1}{2}, 1\right)$$

CASI PARTICOLARI (omogenei)

1) semicirconferenza, massa m, raggio R



$I_{33} = mR^2$

Oy asse di simm.

$\Downarrow$

$Oxyz$ princ. d'inerzia

$I_{11} + I_{22} = I_{33}$

$$I_{11} = \int_F \rho (R\sin\theta)^2 R\,d\theta = \frac{m}{\pi R} R^3 \int_0^{\pi} \sin^2\theta\,d\theta$$

ma $\cos 2\theta = \cos^2\theta - \sin^2\theta = 1 - 2\sin^2\theta$

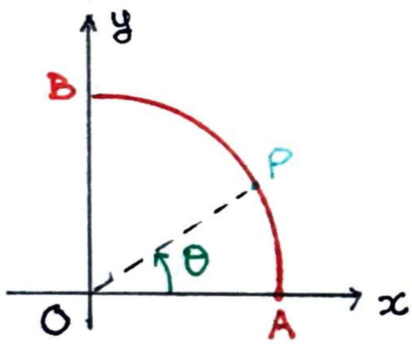
$$\Rightarrow \sin^2\theta = \frac{1-\cos 2\theta}{2}$$

$$\int_0^{\pi} \frac{(1-\cos 2\theta)}{2}\, d\theta = \frac{1}{2}\left[\theta - \frac{1}{2}\sin 2\theta\right]_0^{\pi} = \frac{\pi}{2}$$

Pertanto

$$I_{11} = \frac{m}{\pi} R^2 \cdot \frac{\pi}{2} = \frac{mR^2}{2} \Rightarrow I_{22} = \frac{mR^2}{2}$$

$$\underset{\sim}{I}_O = mR^2 \,\mathrm{diag}\left(\frac{1}{2}, \frac{1}{2}, 1\right)$$



2) 1/4 circonferenza

massa m, raggio R.

$I_{33} = mR^2$

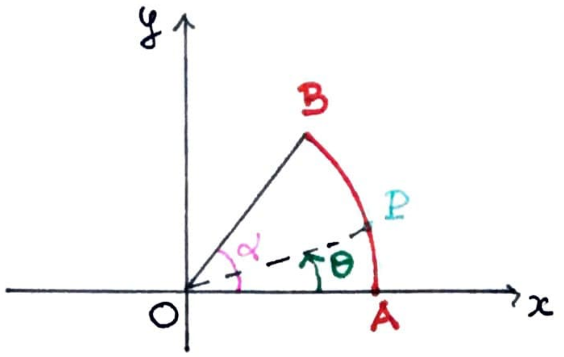
$Oxyz$ non è ad assi principali d'inerzia $\Rightarrow I_{12} \neq 0$

$$\text{ma } I_{11} = I_{22} = \frac{1}{2} I_{33} = \frac{mR^2}{2}$$

$$I_{12} = -\int_F \rho\, xy\, dF = -\frac{m\,2}{\pi R}\int_0^{\pi/2} R\cos\theta \cdot R\sin\theta\, R\, d\theta$$

$$= -\frac{2m}{\pi} R^2 \left(\frac{1}{2}\sin^2\theta\right)_0^{\pi/2} = -\frac{mR^2}{\pi}$$

$$\underset{\sim}{I}_O = \begin{bmatrix} \frac{mR^2}{2} & -\frac{mR^2}{\pi} & 0 \\ -\frac{mR^2}{\pi} & \frac{mR^2}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



3) arco di circonferenza

massa m, raggio R, apertura α.

$$I_{33} = mR^2$$

$$I_{11} = \int_F \rho\, y^2\, dF = \frac{m}{\alpha R} \int_0^\alpha R^2 \sin^2\theta\, R\, d\theta = \frac{mR^2}{\alpha} \int_0^\alpha \sin^2\theta\, d\theta$$

$$= \frac{mR^2}{\alpha} \cdot \frac{1}{2} \left[\theta - \frac{1}{2} \sin 2\theta \right]_0^\alpha = \frac{mR^2}{2\alpha} \left(\alpha - \frac{1}{2} \sin 2\alpha \right)$$

$$= \frac{mR^2}{2} \left(1 - \frac{\sin 2\alpha}{2\alpha} \right)$$

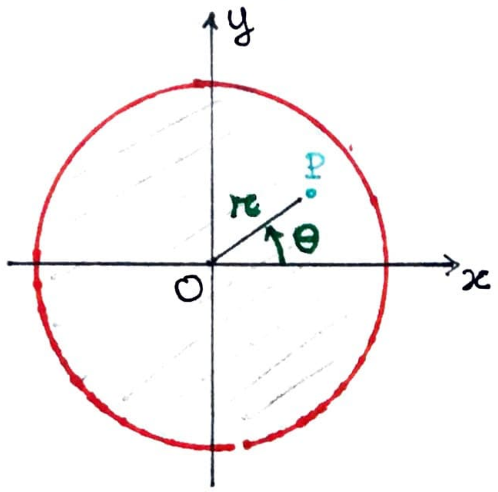
$$I_{22} = I_{33} - I_{11} = mR^2 - \frac{mR^2}{2} + \frac{mR^2}{2} \frac{\sin 2\alpha}{2\alpha} = \frac{mR^2}{2} \left(1 + \frac{\sin 2\alpha}{2\alpha} \right)$$

$$I_{12} = -\frac{m}{\alpha R} \int_0^\alpha R \sin\theta\, R \cos\theta\, R\, d\theta = -\frac{m}{\alpha} R^2 \int_0^\alpha \sin\theta \cos\theta\, d\theta$$

$$= -\frac{mR^2}{\alpha} \left(\frac{1}{2} \sin^2\theta \right)_0^\alpha = -\frac{mR^2}{2\alpha} \sin^2\alpha$$

$$= -\frac{mR^2}{2\alpha} \left(\frac{1 - \cos 2\alpha}{2} \right) = -\frac{mR^2}{2} \left(\frac{1 - \cos 2\alpha}{2\alpha} \right)$$

DISCO



omogeneo, massa m, raggio R.

Oxyz principale d'inerzia

$$I_{11} = I_{22} = \frac{1}{2} I_{33}$$

$$I_{33} = \int_F \rho x^2 \, dF = \int_0^R \int_0^{2\pi} \frac{m}{\pi R^2} \cdot r^2 \, r \, dr \, d\theta$$

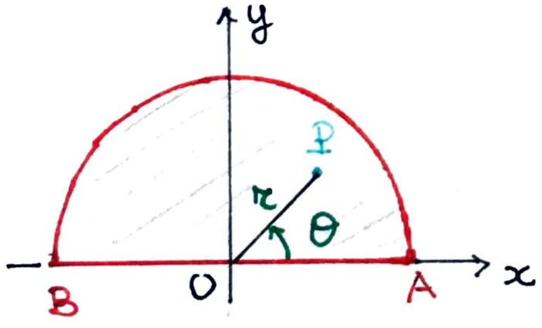
$$= \frac{m}{\pi R^2} \cdot \frac{1}{4} R^4 \cdot 2\pi = \frac{mR^2}{2}$$

$$\Rightarrow \underset{\sim}{I}_O = \frac{mR^2}{2} \operatorname{diag}\left(\frac{1}{2}, \frac{1}{2}, 1\right)$$

Osservazione

Come nel caso della circonferenza $I_{33} = \frac{mR^2}{2}$ per ogni settore circolare di ampiezza α : $\alpha \in (0, 2\pi]$.

CASI PARTICOLARI (omogenei)



① **semidisco** massa m, raggio R.

Oy asse di simmetria $\Rightarrow I_{12} = 0$

Oxyz principale d'inerzia.

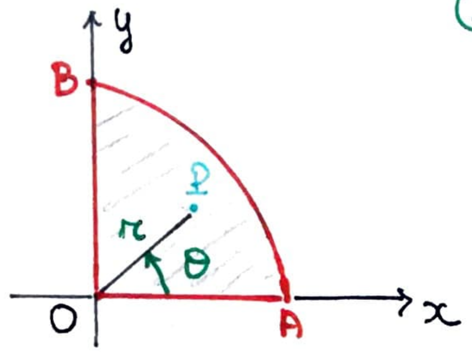
$$I_{11} = \int_F \rho \, y^2 \, dF = \int_0^R \int_0^{\pi} \frac{m\,2}{\pi R^2} \cdot r^2 \sin^2\theta \, r \, dr \, d\theta$$

$$= \frac{2m}{\pi R^2} \cdot \frac{1}{4} R^4 \underbrace{\int_0^{\pi} \sin^2\theta \, d\theta}_{= \pi/2} = \frac{mR^2}{4}$$

$$I_{22} = I_{11} = \frac{mR^2}{4}$$

$$I_O = \frac{mR^2}{2} \operatorname{diag}\left(\frac{1}{2}, \frac{1}{2}, 1\right)$$

② 1/4 disco massa m, raggio R.



$$I_{11} = I_{22} = \frac{1}{2} I_{33} = \frac{mR^2}{4}$$

$$I_{12} \neq 0$$

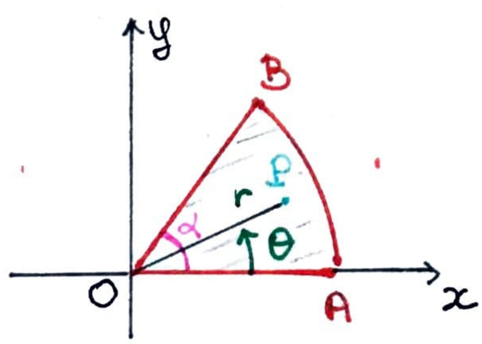
$$I_{12} = -\int_0^{\pi/2}\int_0^R \frac{m\,4}{\pi R^2}\, r\sin\theta\, r\cos\theta\, r\,dr\,d\theta$$

$$= -\frac{4m}{\pi R^2}\cdot\frac{1}{4}R^4 \underbrace{\int_0^{\pi/2}\sin\theta\cos\theta\,d\theta}_{=\frac{1}{2}} = -\frac{mR^2}{2\pi}$$

$$\underset{\sim}{I}_O = \begin{bmatrix} \frac{mR^2}{4} & -\frac{mR^2}{2\pi} & 0 \\ -\frac{mR^2}{2\pi} & \frac{mR^2}{4} & 0 \\ 0 & 0 & \frac{mR^2}{2} \end{bmatrix}$$

③ settore circolare

massa m, raggio R, apertura α.



$$I_{33} = \frac{mR^2}{2}$$

$$A = \frac{\alpha R\cdot R}{2} = \frac{\alpha R^2}{2}$$

$$I_{12} \neq 0.$$

$$I_{12} = -\int_0^R\int_0^\alpha \frac{m\,2}{\alpha R^2}\cdot r\sin\theta\, r\cos\theta\, r\,dr\,d\theta$$

$$= -\frac{m}{\alpha R^2}2\cdot\frac{1}{4}R^4 \underbrace{\int_0^\alpha \sin\theta\cos\theta\,d\theta}_{=\frac{1}{2}\sin^2\alpha}$$

$$= -\frac{mR^2}{4\alpha}\sin^2\alpha$$

$$I_{11} = \int_0^R \int_0^\alpha \frac{m \cdot 2}{\alpha R^2} \cdot r^2 \sin^2\theta \, r \, dr \, d\theta$$

$$= \frac{2m}{\alpha R^2} \cdot \frac{1}{4} R^4 \int_0^\alpha \sin^2\theta \, d\theta = \frac{mR^2}{2\alpha} \cdot \frac{1}{2} \left(\alpha - \frac{1}{2} \sin 2\alpha\right)$$

$$= \frac{mR^2}{4} \left(\frac{2\alpha - \sin 2\alpha}{2\alpha}\right)$$

$$I_{22} = I_{33} - I_{11} = \frac{mR^2}{4} \left(\frac{2\alpha + \sin 2\alpha}{2\alpha}\right)$$