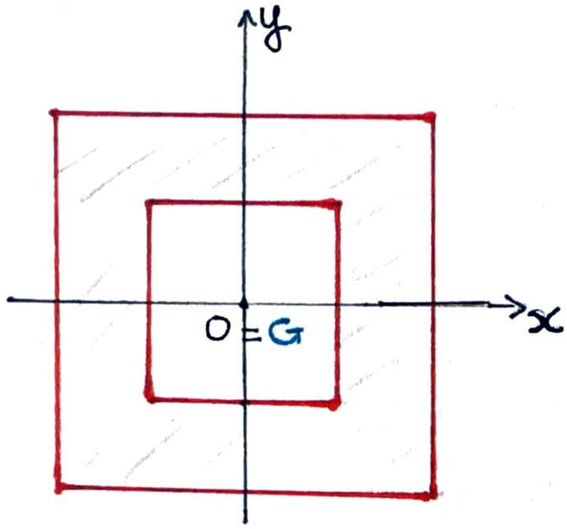
LAMINE CON FORI



① lamina quadrata con foro quadrato concentrico.

omogenea, massa m

lati L, ℓ.

$Oxyz$ principale d'inerzia

$I_{11} = I_{22} = \frac{1}{2} I_{33}$

Pieno: lamina di lato L e massa m_1

Vuoto: lamina di lato ℓ e massa m_2

$$m = m_1 - m_2 \quad , \quad m = \rho A = \rho (L^2 - \ell^2) \Rightarrow \begin{cases} m_1 = \dfrac{m L^2}{(L^2 - \ell^2)} \\ m_2 = \dfrac{m \ell^2}{(L^2 - \ell^2)} \end{cases}$$

$$\underset{\sim}{I}_O^P = m_1 L^2/_{12} \operatorname{diag}(1, 1, 2)$$

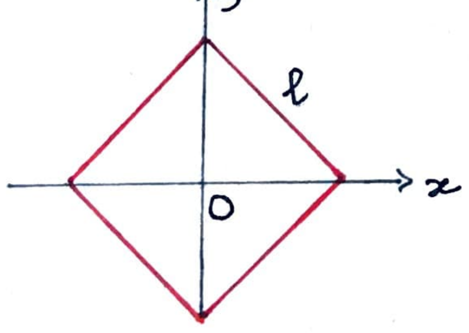
$$\underset{\sim}{I}_O^V = m_2 \ell^2/_{12} \operatorname{diag}(1, 1, 2)$$

$$\underset{\sim}{I}_O = \frac{1}{12} \operatorname{diag}(1, 1, 2) \, (m_1 L^2 - m_2 \ell^2)$$

$$m_1 L^2 - m_2 \ell^2 = m \frac{(L^4 - \ell^4)}{(L^2 - \ell^2)} = m (L^2 + \ell^2)$$

$$= \frac{m (\ell^2 + L^2)}{12} \operatorname{diag}(1, 1, 2)$$

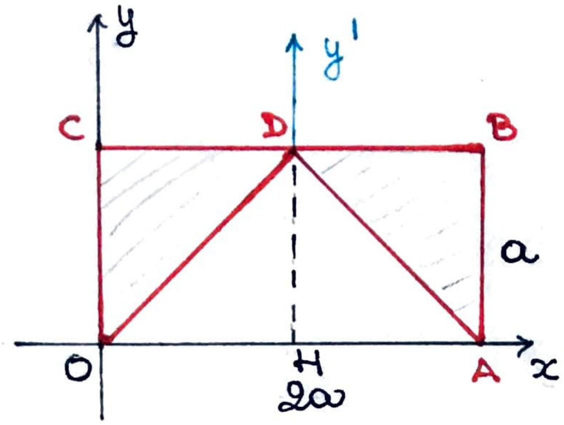
se il quadrato di lato ℓ viene ruotato di $\pi/2$



$$I_{33} = \frac{1}{6} m_2 \ell^2$$

$$I_{11} = I_{22} = \frac{1}{12} m_2 \ell^2$$

$\Rightarrow \underset{\sim}{I}_O$ è invariato



② lamina rettangolare con foro triangolare (rettangolo isoscele)

omogenea, massa m, lati a, 2a

Oxyz <u>non</u> è princip. d'inerzia

Rettangolo massa $m_1 = \rho A_1 = \rho 2a^2$

Triangolo rett. isoscele $m_2 = \rho A_2 = \rho a^2$ $\Rightarrow m = \rho a^2$

$$\Rightarrow \begin{cases} m_1 = 2m \\ m_2 = m \end{cases}$$

Rettangolo : $I_{11} = \frac{1}{3} m_1 a^2$; $I_{22} = \frac{1}{3} m_1 \cdot 4a^2$, $I_{33} = \frac{1}{3} m_1 5a^2$

$$I_{12} = -m_1 a \cdot \frac{a}{2} = -\frac{m_1 a^2}{2} \qquad G_1\left(a, \frac{a}{2}\right)$$

Triangolo : $I_{11} = \frac{1}{6} m_2 h^2 = \frac{1}{6} m_2 a^2$

$$I_{22} = I'_{22} + m_2 d^2 \quad \text{dove } d = a$$

$$I'_{22} = \frac{1}{24} m_2 4a^2 = \frac{m_2 a^2}{6}$$

oppure $I'_{22} = 2 \cdot \frac{1}{6} \frac{m_2}{2} a^2$

$$\Rightarrow I_{22} = \frac{7}{6} m_2 a^2$$

$$I_{33} = \frac{4}{3} m_2 a^2$$

$$I_{12} = -m_2 a \cdot \frac{a}{3} = -m_2 \frac{a^2}{3} \qquad G_2\left(a, \frac{a}{3}\right)$$

Sottraendo termini corrispondenti si ha:

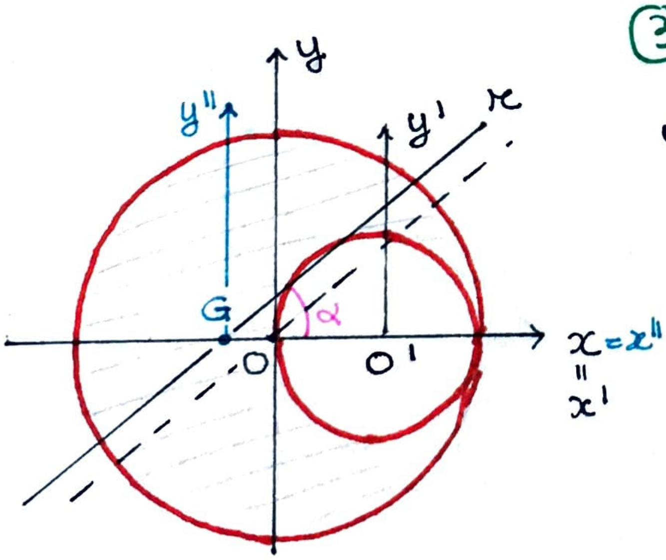
$$I_{11}^{TOT} = \frac{1}{3} \cdot 2ma^2 - \frac{1}{6} ma^2 = \frac{1}{2} ma^2$$

$$I_{22}^{TOT} = \frac{1}{3} \cdot 2m \cdot 4a^2 - \frac{7}{6} ma^2 = \frac{3}{2} ma^2$$

$$I_{33}^{TOT} = 2ma^2$$

$$I_{12}^{TOT} = -ma^2 + \frac{1}{3} ma^2 = -\frac{2}{3} ma^2$$

$$\underset{\sim}{I}_O = ma^2 \begin{bmatrix} \frac{1}{2} & -\frac{2}{3} & 0 \\ -\frac{2}{3} & \frac{3}{2} & 0 \\ 0 & 0 & 2 \end{bmatrix}$$



③ disco con foro circolare

omogeneo, massa m

raggi R, $\frac{R}{2}$.

Asse Ox è asse di simmetria

$Oxyz$ è principale d'inerzia

Disco raggio R $\quad m_1 = \rho \pi R^2$

Disco raggio $\frac{R}{2}$ $\quad m_2 = \rho \pi \frac{R^2}{4}$

$$m = \rho \cdot \frac{3}{4} \pi R^2$$

$m_1 = \frac{4}{3} m \; ; \; m_2 = \frac{m}{3}$

$\overset{1}{\text{Disco}} \Rightarrow \underset{\sim}{I}_O^{1} = m_1 \frac{R^2}{4} \text{diag}(1,1,2) = \frac{mR^2}{3} \text{diag}(1,1,2)$

Disco $\Rightarrow$ $\check{\underset{\sim}{I}}_0{}' = m_2 \frac{1}{4}\left(\frac{R}{2}\right)^2 \mathrm{diag}(1,1,2) = \frac{m}{48} R^2 \mathrm{diag}(1,1,2)$

Ora devo traslare in O: con Huygens:

$$\check{I}_{22} = I_{22}' + m_2 \frac{R^2}{4} = \frac{1}{4} m_2 \frac{R^2}{4} + m_2 \frac{R^2}{4} = \frac{5}{4} m_2 \frac{R^2}{4} = \frac{5}{48} m R^2$$

$$\check{I}_{11} = I_{11}'$$

$$\check{I}_{33} = \frac{mR^2}{48} + \frac{5}{48} mR^2 = \frac{1}{8} mR^2$$

$$\Rightarrow \begin{cases} I_{11}^{TOT} = \frac{mR^2}{3} - \frac{mR^2}{48} = \frac{15}{48} mR^2 = \frac{5}{16} mR^2 \\ I_{22}^{TOT} = \frac{mR^2}{3} - \frac{5}{48} mR^2 = \frac{11}{48} mR^2 \\ I_{33}^{TOT} = \frac{13}{24} mR^2 \end{cases}$$

Calcoliamo $\underset{\sim}{I}_G$ dove G è il baricentro $x_G = -\frac{R}{6}$

$$\begin{cases} I_{11}^G = I_{11} = \frac{5}{16} mR^2 \\ I_{22}^G = I_{22} - m\left(\frac{R}{6}\right)^2 = \frac{11}{48} mR^2 - \frac{mR^2}{36} = \frac{29}{144} mR^2 \\ I_{33}^G = \frac{37}{72} mR^2 \end{cases}$$

Vogliamo ora calcolare il momento $I_{G\alpha}$ dove $\bar{\alpha}$ versore della retta r, parallela ad $y = x$, passante per G.

$I_{G\alpha} = \underset{\sim}{I}_G \bar{\alpha} \cdot \bar{\alpha}$ $\quad \bar{\alpha} = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0\right)$ $\quad \alpha = \pi/4$

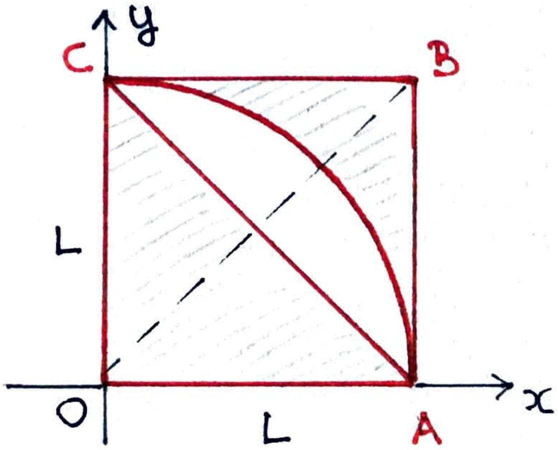
$$= \frac{1}{2} I_{33} = \frac{37}{144} mR^2$$

Esercizio proposto

Data la lamina omogenea, di massa m, descritta in figura, calcolare:

1) baricentro

2) matrice d'inerzia rispetto ad O.



Soluzioni:

1) coordinate di G $\left(\frac{4L}{3(6-\pi)} ; \frac{4L}{3(6-\pi)} \right)$

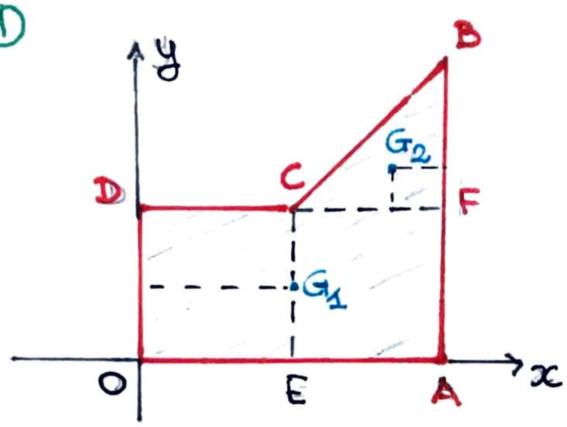
2) $I_{11} = I_{22} = \frac{(20-3\pi)}{12(6-\pi)} m L^2$

$$I_{33} = \frac{(20-3\pi)}{6(6-\pi)} m L^2$$

$$I_{12} = -\frac{2 m L^2}{3(6-\pi)}$$

Esercizi tipo esame

①



Lamina omogenea

massa m

$\overline{OA} = \overline{AB} = 2L$

$\overline{OD} = \overline{DC} = L$

ogni triangolino ha massa $\frac{m}{5}$

Scompongo per esempio in $OAFD + C\hat{F}B$

Rettangolo $m_R = \frac{4}{5}m$

$$I_{11} = \frac{1}{3} m_R L^2 = \frac{4}{15} m L^2$$

$$I_{22} = \frac{1}{3} m_R \cdot 4L^2 = \frac{16}{15} m L^2$$

$$I_{33} = \frac{1}{3} m_R \, 5L^2 = \frac{4}{3} m L^2$$

$$I_{12} = - m_R \, L \cdot \frac{L}{2} = -\frac{1}{2} m_R L^2 = -\frac{2}{5} m L^2$$

Triangolo $m_T = \frac{m}{5}$

$$I_{11} = I_{11}^{G_2} + m_T \, y_{G_2}^2 \qquad y_{G_2} = L + \frac{L}{3} = \frac{4}{3} L$$

$$I_{11}^{G_2} = I_{CF} - m_T \frac{L^2}{9} = \frac{m_T L^2}{6} - m_T \frac{L^2}{9} = \frac{1}{18} m_T L^2$$

$$\Rightarrow I_{11} = \frac{1}{18} m_T L^2 + m_T \cdot \frac{16}{9} L^2 = \frac{33}{18} m_T L^2 = \frac{33}{90} m L^2 = \frac{11}{30} m L^2$$

$$I_{22} = I_{22}^{G_2} + m_T \, x_{G_2}^2 \qquad x_{G_2} = L + \frac{2}{3} L = \frac{5}{3} L$$

$$I_{22}^{G_2} = I_{BF} - m_T \frac{L^2}{9} = \frac{m_T L^2}{6} - m_T \frac{L^2}{9} = \frac{1}{18} m_T L^2$$

$$I_{22} = \frac{1}{18} m_T L^2 + \frac{25}{9} m_T L^2 = \frac{51}{18} m_T L^2 = \frac{51}{90} m L^2 = \frac{17}{30} m L^2$$

$$I_{33} = \frac{28}{30} m L^2 = \frac{14}{15} m L^2$$

$$I_{12} = I_{12}^{G_2} - m_T \underbrace{x_{G_2} y_{G_2}}_{\frac{20}{9}L^2} = I_{12}^{G} - \frac{4}{9} m L^2$$

$$\underset{?}{\downarrow} \qquad\qquad \underset{?}{\downarrow}$$

$$I_{12} = -\int_F \rho x y \, dF = -\int_L^{2L} \int_L^{x} \rho x y \, dx \, dy$$

$$= - m_T \frac{2}{L^2} \int_{+L}^{2L} x \left(\int_L^x y \, dy \right) dx$$

$$= - \frac{2}{L^2} m_T \int_{+L}^{2L} x \cdot \frac{1}{2} (x^2 - L^2) \, dx$$

$$= - \frac{m_T}{L^2} \int_{+L}^{2L} (x^3 - L^2 x) \, dx = - \frac{m_T}{L^2} \left(\frac{1}{4} x^4 - L^2 \cdot \frac{1}{2} x^2 \right)_{+L}^{2L}$$

$$= - \frac{m_T}{L^2} \left[\frac{1}{4} \cdot 16 L^4 - L^2 \cdot \frac{1}{2} 4 L^2 - \left(\frac{1}{4} L^4 - \frac{1}{2} L^4 \right) \right]$$

$$= - \frac{m_T}{L^2} \left(4L^4 - 2L^4 - \frac{1}{4} L^4 + \frac{1}{2} L^4 \right)$$

$$= - m_T L^2 \left(2 - \frac{1}{4} + \frac{1}{2} \right) = - \frac{9}{4} m_T L^2 = - \frac{9}{20} m L^2$$

$$\left(\Rightarrow \quad I_{12}^{G} = - \frac{9}{20} m L^2 + \frac{4}{9} m L^2 = \frac{-81 + 80}{180} m L^2 = - \frac{1}{180} m L^2 \right)$$

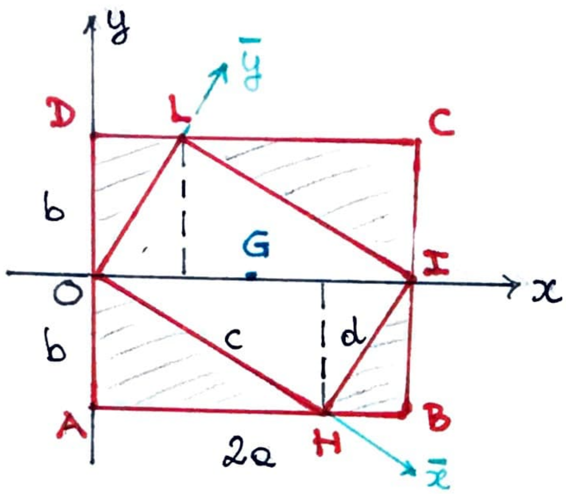
$$I_{11}^{TOT} = \left(\frac{4}{15} + \frac{11}{30} \right) m L^2 = \frac{19}{30} m L^2$$

$$I_{22}^{TOT} = \left(\frac{16}{15} + \frac{17}{30} \right) m L^2 = \frac{49}{30} m L^2$$

$$I_{33}^{TOT} = \frac{34}{15} m L^2$$

$$I_{12}^{TOT} = - \left(\frac{2}{5} + \frac{9}{20} \right) m L^2 = - \frac{17}{20} m L^2$$

② Calcolare i momenti I_{11}, I_{22}, I_{33} di:



Lamina omogenea rettangolare con foro rettangolare: m

$m_{pieno} = 2m$

$m_{vuoto} = m$

$\overline{AB} = 2a$ $\quad \overline{OH} = c$

$\overline{AD} = 2b$ $\quad \overline{OL} = d$

- Rettangolo ABCD

$$I_{11} = \frac{1}{12} \cdot (2m)(2b)^2 = \frac{2}{3} m b^2$$

$$I_{22} = \frac{1}{3}(2m)(2a)^2 = \frac{8}{3} m a^2$$

$$I_{33} = \frac{m}{3}(8a^2 + 2b^2)$$

($I_{12} = 0$ $\quad$ Ox è asse di simmetria)

- Rettangolo OHIL

$$I_{11} = 2\, I_{11}(\text{triangolo OHI}) = 2 \cdot \frac{1}{6}\left(\frac{m}{2}\right) b^2 = \frac{1}{6} m b^2$$

$$I_{33} = \bar{I}_{33} = \bar{I}_{11} + \bar{I}_{22} = \frac{1}{3} m d^2 + \frac{1}{3} m c^2 = \frac{1}{3} m (c^2 + d^2)$$

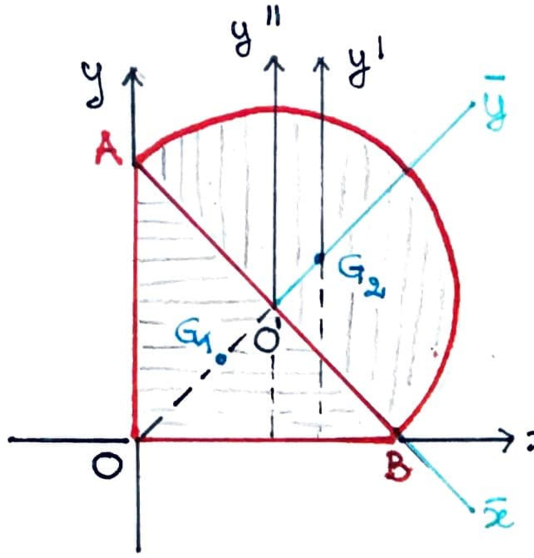
ma $c^2 + d^2 = 4a^2$

$$\Rightarrow I_{33} = \frac{1}{3} m\, 4a^2 = \frac{4}{3} m a^2$$

$$\Rightarrow I_{22} = I_{33} - I_{11} = \frac{4}{3} m a^2 - \frac{1}{6} m b^2 = \frac{1}{6} m (\underbrace{8a^2 - b^2}_{>0})$$

- $I_{11}^{TOT} = \frac{2}{3} m b^2 - \frac{1}{6} m b^2 = \frac{3}{6} m b^2 = \frac{1}{2} m b^2$
- $I_{22}^{TOT} = \frac{8}{3} m a^2 - \frac{8}{6} m a^2 + \frac{m b^2}{6} = \frac{m}{6}(8a^2 + b^2)$
- $I_{33}^{TOT} = \frac{m}{3}(8a^2 + 2b^2) - \frac{4}{3} m a^2 = \frac{m}{3}(4a^2 + 2b^2)$

③ Calcolare I_{22} della seguente figura:



- semidisco omogeneo di massa m e raggio R
- triangolo rettangolo isoscele omogeneo di massa m

$\Downarrow$

$\overline{OB} = R\sqrt{2}$

semidisco e triangolo hanno densità di massa DIVERSE

- Triangolo $A\hat{O}B$

$$I_{22} = \frac{1}{6} m (R\sqrt{2})^2 = \frac{1}{3} m R^2$$

- Semidisco

$$I_{22} = I_{22}^{G_2} + m d^2 \qquad d = x_{G_2}$$

$$\overline{O'G_2} = \frac{4R}{3\pi}$$

$$x_{G_2} = x_{O'} + \frac{4R}{3\pi} \cdot \frac{\sqrt{2}}{2} = R\frac{\sqrt{2}}{2} + R\frac{\sqrt{2}}{2}\frac{4}{3\pi} = R\frac{\sqrt{2}}{2}\left(1 + \frac{4}{3\pi}\right)$$

$$I_{22}^{G_2} = I_{22}^{O'} - m d'^2 \qquad d' = \overline{O'G_2}\frac{\sqrt{2}}{2} = \frac{4R}{3\pi}\frac{\sqrt{2}}{2}$$

$$I_{22}^{O'} = \bar{I}_{O'} \bar{J}'' \cdot \bar{J}'' \quad \text{dove } \bar{J}'' = -\frac{\sqrt{2}}{2}\bar{e}_1 + \frac{\sqrt{2}}{2}\bar{e}_2$$

$(\bar{e}_1, \bar{e}_2, \bar{e}_3)$ base di $O'\bar{x}\bar{y}\bar{z}$

$$\underset{\sim}{\bar{I}}_{O'} = \mathrm{diag}\left(\frac{1}{4}, \frac{1}{4}, \frac{1}{2}\right) m R^2$$

$$\Rightarrow I_{22}^{O'} = \frac{1}{4} m R^2$$

$$I_{22}^{G_2} = \frac{1}{4} m R^2 - m \cdot \frac{8}{9}\frac{R^2}{\pi^2}$$

$$I_{22} = \frac{1}{4} m R^2 - m \cdot \frac{8}{9}\frac{R^2}{\pi^2} + m \cdot \frac{R^2}{2}\left(1 + \frac{4}{3\pi}\right)^2$$

$$I_{22} = mR^2\left(\frac{1}{2} + \cancel{\frac{8}{9\pi^2}} + \frac{4}{3\pi} - \cancel{\frac{8}{9\pi^2}} + \frac{1}{4}\right)$$

$$= mR^2\left(\frac{3}{4} + \frac{4}{3\pi}\right)$$

$$I_{22}^{TOT} = \frac{1}{3}mR^2 + mR^2\left(\frac{3}{4} + \frac{4}{3\pi}\right)$$

$$= mR^2\,\frac{(13\pi + 16)}{12\pi}$$