Esercizi sui sistemi di vettori applicati

1) Calcolare l'equazione cartesiana dell'a.c. del seguente Σ_a:

$A_1(1,0,0)$ $\quad A_2(0,0,2)$ $\quad A_3(1,1,0)$

$\bar{v}_1(1,1,0)$ $\quad \bar{v}_2(0,0,1)$ $\quad \bar{v}_3\left(\frac{1}{2},-\frac{1}{2},0\right)$

$$\bar{R}=\sum_{i=1}^{3}\bar{v}_i=\left(\frac{3}{2},\frac{1}{2},1\right)$$

$$\bar{M}_O=\sum_{i=1}^{3}(A_i-O)\times\bar{v}_i=(1,0,0)\times(1,1,0)+\underbrace{(0,0,2)\times(0,0,1)}_{=\bar{0}}$$

$$+(1,1,0)\times\left(\frac{1}{2},-\frac{1}{2},0\right)=(0,0,1)+(0,0,-1)=\bar{0}$$

$$\Rightarrow I=\bar{R}\cdot\bar{M}_O\equiv 0 \quad e \quad \bar{R}\times\bar{M}_O\equiv\bar{0}$$

$$\boxed{O'-O=\frac{\bar{R}\times\bar{M}_O}{R^2}+\lambda\bar{R}}\boxed{=\lambda(O')\bar{R}}$$ eq. dell'asse centrale

$$(x,y,z)=\lambda\left(\frac{3}{2},\frac{1}{2},1\right)$$

$$\begin{cases} x=\frac{3}{2}\lambda \\ y=\frac{1}{2}\lambda \\ z=\lambda \end{cases}$$ eq. parametrica $\Rightarrow$ $\boxed{\frac{2}{3}x=2y=z}$

eq. cartesiana dell'A.C.

2) Dire a cosa è equivalente il seguente Σ_a:

$A_1(1,0,1)$ $\quad A_2(0,1,1)$ $\quad A_3(1,1,0)$

$\bar{v}_1(1,-1,0)$ $\quad \bar{v}_2(0,1,-1)$ $\quad \bar{v}_3(-1,0,1)$

$$\bar{R}=\sum_{i=1}^{3}\bar{v}_i=(0,0,0)\equiv\bar{0}$$

Allora $\bar{M}_O$ non dipende dal polo O.

Scegliamo $O \equiv A_1$.

$$\bar{M}_{A_1} = \sum_{i=1}^{3} (A_i - A_1) \times \bar{v}_i = (A_2 - A_1) \times \bar{v}_2 + (A_3 - A_1) \times \bar{v}_3 =$$

$$= (-1, 1, 0) \times (0, 1, -1) + (0, 1, -1) \times (-1, 0, 1)$$

$$= (-1, -1, -1) + (1, 1, 1) \equiv \vec{0}$$

Allora $\Sigma_{i_2} \simeq$ zero, cioè una coppia di braccio nullo.

3) Stabilire quale dei punti indicati appartiene all'a.c. del seguente Σ_{ia}:

$A_1\ (1, 0, 0)$ $\qquad A_2\ (0, 1, 0)$ $\qquad A_3\ (0, 0, 1)$

$\bar{v}_1\ (1, 0, 0)$ $\qquad \bar{v}_2\ (0, 2, 0)$ $\qquad \bar{v}_3\ (0, -1, 1)$

<u>Punti</u>:

$A = \left(0, \frac{1}{3}, \frac{1}{3}\right)$; $B = \left(1, \frac{2}{3}, \frac{4}{3}\right)$; $C = \left(0, \frac{1}{3}, \frac{2}{3}\right)$; $D = \left(\frac{1}{3}, \frac{2}{3}, 0\right.$

$\bar{R} = (1, 1, 1)$ e $R^2 = 3$

$$\bar{M}_O = \sum_{i=1}^{3} (A_i - O) \times \bar{v}_i = \underbrace{\cancel{(1,0,0) \times (1,0,0)}}_{=\bar{0}} + \underbrace{\cancel{(0,1,0) \times (0,2,0}}_{=\bar{0}}$$

$$+ (0, 0, 1) \times (0, -1, 1) = (1, 0, 0)$$

$I = \bar{R} \cdot \bar{M}_O = 1$

$\bar{R} \times \bar{M}_O = (1, 1, 1) \times (1, 0, 0) = (0, 1, -1)$

$$\begin{cases} x = 0 + \lambda \\ y = \frac{1}{3} + \lambda \\ z = -\frac{1}{3} + \lambda \end{cases} \Rightarrow x = y - \frac{1}{3} = z + \frac{1}{3} \Rightarrow D \in A.C.$$

VA2

4) Stabilire la massima riduzione del seguente Σ_a:

$A_1(1,0,0)$ $\quad A_2(0,1,0)$ $\quad A_3(0,0,1)$

$\bar{v}_1(0,0,1)$ $\quad \bar{v}_2(-1,0,-1)$ $\quad \bar{v}_3(1,-1,0)$

$\bar{R}=(0,-1,0)$

$$\bar{M}_0=(1,0,0)\times(0,0,1)+(0,1,0)\times(-1,0,-1)+$$
$$+(0,0,1)\times(1,-1,0)=(0,-1,0)+(-1,0,1)+$$
$$+(1,1,0)=(0,0,1)$$

$I=\bar{R}\cdot\bar{M}_0=0$

TEOREMA 1

se $I=0$ con $\bar{R}\neq\bar{0}$ $\exists$ un punto $O'\in$ a.c. tale che

$\bar{M}_{O'}=\vec{0} \Rightarrow \Sigma_a \simeq (O',\bar{R})$

Quindi Σ_a è riducibile ad un vettore applicato (in un punto dell'asse centrale)

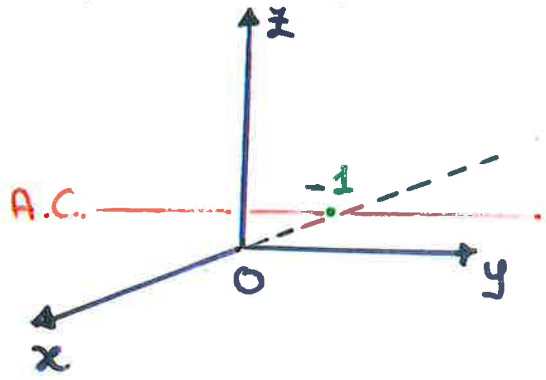
Dalla L.V.M:

$$\boxed{\bar{M}_{O'}=\bar{M}_0+\bar{R}\times(O'-O)} \qquad O'=(x,y,z)$$

$$=(0,0,1)+(0,-1,0)\times(x,y,z)$$
$$=(0,0,1)+(-z,0,x)=(-z,0,x+1)$$

Ma $\bar{M}_{O'}\equiv\bar{0}$

$$\Rightarrow \begin{cases} z=0 \\ x=-1 \\ y \text{ qualunque} \end{cases} \Rightarrow O'=(-1,\lambda,0) \quad \lambda\in\mathbb{R}$$



VA3

5) Determinare il centro del seguente Σ_p (e concordi):

$A_1(1,0,0)$ $\quad A_2(0,1,0)$ $\quad A_3(0,0,1)$

$\bar{v}_1\left(1,\frac{1}{2},\frac{1}{3}\right)$ $\quad \bar{v}_2\left(2,1,\frac{2}{3}\right)$ $\quad \bar{v}_3\left(3,\frac{3}{2},1\right)$

Posto $\bar{a}=\left(1,\frac{1}{2},\frac{1}{3}\right)$ si ha che $\begin{cases} \bar{v}_1=\bar{a} & \left(=\frac{7}{6}\vec{u}\right) \\ \bar{v}_2=2\bar{a} & \left(=\frac{7}{3}\bar{u}\right) \\ \bar{v}_3=3\bar{a} & \left(=\frac{7}{2}\bar{u}\right) \end{cases}$

$$\bar{R}=\sum_{i=1}^{3}\bar{v}_i=\sum_{i=1}^{3}v_i\,\bar{a}=6\bar{a}\left(=6\,|\bar{a}|\,\vec{u}=7\vec{u}\right)$$

dove $|\bar{a}|=\sqrt{1+\frac{1}{4}+\frac{1}{9}}=\sqrt{\frac{49}{36}}=\frac{7}{6}$

$\bar{a}$ è la direzione comune, il cui versore $\vec{u}=\frac{\vec{a}}{|\bar{a}|}$

$$\boxed{C-O=\frac{\sum_{i=1}^{3}v_i(A_i-O)}{\sum_{i=1}^{3}v_i}}=\frac{1}{6}\left[1\cdot(1,0,0)+2\cdot(0,1,0)+3\cdot(0,0,1)\right]$$

$$=\frac{1}{6}\left[(1,0,0)+(0,2,0)+(0,0,3)\right]=\frac{1}{6}(1,2,3)$$

$\Rightarrow$

$$C=\left(\frac{1}{6},\frac{1}{3},\frac{1}{2}\right)$$

6) Dato il seguente Σ_a:

$A_1(1,0,0)$ $\quad A_2(0,0,2)$ $\quad A_3(0,1,0)$

$\bar{v}_1(1,1,0)$ $\quad \bar{v}_2(0,0,1)$ $\quad \bar{v}_3\left(\frac{1}{2},-\frac{1}{2},0\right)$

calcolare il modulo del momento di Σ_a rispetto ai punti dell'a.c.

$\bar{R}=\left(\frac{3}{2},\frac{1}{2},1\right)$ e $|\bar{R}|=\sqrt{\frac{9}{4}+\frac{1}{4}+1}=\sqrt{\frac{7}{2}}$

$$\bar{M}_o = (1,0,0)\times(1,1,0) + \underbrace{\cancel{(0,0,2)\times(0,0,1)}}_{=\bar{0}} + (0,1,0)\times\left(\tfrac{1}{2}, -\tfrac{1}{2}, \ldots\right.$$

$$= (0,0,1) + \left(0,0,-\tfrac{1}{2}\right) = \left(0,0,\tfrac{1}{2}\right)$$

$$I = \bar{R}\cdot\bar{M}_o = \left(\tfrac{3}{2}, \tfrac{1}{2}, 1\right)\cdot\left(0,0,\tfrac{1}{2}\right) = \tfrac{1}{2}$$

I punti dell'a.c. son quelli che hanno momento di modulo minimo poichè per essi $\bar{M}_{O'} \parallel \bar{R}$.

$$\boxed{\bar{M}_{O'} = \frac{I}{R^2}\vec{R} = \frac{I}{R}\frac{\vec{R}}{|\bar{R}|}}$$

quindi

$$\boxed{|\bar{M}_{O'}| = \frac{I}{R}} = \frac{1}{2}\cdot\sqrt{\frac{2}{7}} = \frac{1}{\sqrt{14}}.$$

7) Dato il seguente Σ_a:

$A_1(1,\alpha,0)$ $\quad A_2(\alpha,0,1)$ $\quad A_3(0,0,\alpha)$

$\bar{v}_1(\alpha,0,1)$ $\quad \bar{v}_2(0,\alpha,1)$ $\quad \bar{v}_3(1,1,0)$

determinare il valore di α affinchè Σ_a abbia $I=-4$.

$$\bar{R} = (\alpha+1, \alpha+1, 2)$$

$$\bar{M}_o = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & \alpha & 0 \\ \alpha & 0 & 1 \end{vmatrix} + \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \alpha & 0 & 1 \\ 0 & \alpha & 1 \end{vmatrix} + \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 0 & \alpha \\ 1 & 1 & 0 \end{vmatrix}$$

$$= (\alpha, -1, -\alpha^2) + (-\alpha, -\alpha, \alpha^2) + (-\alpha, \alpha, 0)$$

$$= (-\alpha, -1, 0)$$

$$I = \bar{R} \cdot \bar{M}_0 = -\alpha(\alpha+1) - (\alpha+1) = -(\alpha+1)^2$$

$$I = -4 \quad \Rightarrow \quad (\alpha+1)^2 = 4$$

$$\Rightarrow \quad \alpha + 1 = \pm 2 \quad \begin{cases} \alpha_1 = 1 \\ \alpha_2 = -3 \end{cases}$$

- $O' \in A.C.$ $\quad \bar{M}_{O'} \begin{cases} \parallel \bar{R} \\ = \bar{0} \end{cases}$

$$\bar{M}_{O'} = \lambda \bar{R}^{0}$$

$$I = \bar{M}_{O} \cdot \bar{R} = \bar{M}_{O'} \cdot \bar{R}^{0} = \lambda \bar{R}^{0} \cdot \bar{R}^{0} = \lambda R^2 \Rightarrow \lambda = \frac{I}{R^2}$$

$$\bar{M}_{O'} = \frac{I}{R^2} \bar{R}^{0} = \frac{I}{R} \frac{\bar{R}^{0}}{R} = \frac{I}{R} \bar{u}^{0}_{R}$$

$$|\bar{M}_{O'}| = \frac{|I|}{R}$$

$\bar{v}_1 = \bar{a}$

$\bar{a} = |\bar{a}|\,\bar{u} = \frac{7}{6}\bar{u}$

$\bar{u} = \frac{\bar{a}}{|\bar{a}|} = \frac{6}{7}\bar{a} = \left(\frac{6}{7}, \frac{3}{7}, \frac{2}{7}\right)$

$|\bar{u}| = \frac{1}{7}\sqrt{36+9+4} = 1$

$\frac{1}{7}\left(\frac{7}{6}, \frac{7}{3}, \frac{7}{2}\right) = \underbrace{\left(\frac{1}{6}, \frac{1}{3}, \frac{1}{2}\right)} = \frac{1}{6}(1, 2, 3)$

$\bar{M}_O = \bar{M}_O^{\parallel} + \bar{M}_O^{\perp} \qquad \parallel \bar{R}, \quad \perp \bar{R}$

$\bar{M}_O^{\parallel} \parallel \bar{R} \qquad \bar{M}_O^{\parallel} = \lambda_0 \bar{R}$

$\bar{M}_O^{\perp} \perp \bar{R} \qquad \bar{M}_O^{\perp} \cdot \bar{R} = 0$

$I = \bar{M}_O \cdot \bar{R} = \bar{M}_O^{\parallel} \cdot \bar{R} + \underbrace{\cancel{\bar{M}_O^{\perp} \cdot \bar{R}}}_{=0} = \lambda_0 \bar{R} \cdot \bar{R} = \lambda_0 R^2$

$\lambda_0 = \frac{I}{R^2}$ ma I non dip. da O $\Rightarrow$ $\cancel{\lambda_0}$

$\lambda = \frac{I}{R^2}$

$\bar{M}_O^{\parallel} = \frac{I}{R^2}\bar{R} = \frac{I}{R}\left(\frac{\bar{R}}{R}\right) = \frac{I}{R}\bar{u}_R$

$|\bar{M}_O^{\parallel}| = \frac{I}{R}$

se $O' \in A.C.$ $\quad \bar{M}_{O'} = \begin{cases} \bar{0} \\ \parallel \bar{R} \end{cases} \Rightarrow |\bar{M}_{O'}| = \frac{I}{R}$
